

ON RATIONAL POINTS OF VARIETIES OVER LOCAL FIELDS HAVING A MODEL WITH TAME QUOTIENT SINGULARITIES

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ABSTRACT. We study rational points on a smooth variety X over a complete local field K with algebraically closed residue field, and models \mathcal{X} of X with tame quotient singularities. If \mathcal{X} is the quotient of a Galois action on a weak Néron model of the base change of X to a tame Galois extension of K , then we construct a canonical weak Néron model of X with a map to \mathcal{X} , and examine its special fiber. As an application we get examples of singular models \mathcal{X} such that there are K -rational points of X specializing to a singular point of \mathcal{X} . Moreover we obtain formulas for the motivic Serre invariant and the rational volume, and the existence of K -rational points on certain K -varieties with potential good reduction.

1. INTRODUCTION

In this article we study smooth and proper varieties over a complete local field K with algebraically closed residue field k with regard to the existence of K -rational points.

A standard way to detect rational points of varieties over complete local fields is to look at models. A model of a K -variety X is an integral, flat scheme \mathcal{X} over the ring of integers \mathcal{O}_K such that the generic fiber of \mathcal{X} is isomorphic to X . There is a natural map $\mathcal{X}(\mathcal{O}_K) \rightarrow X(K)$, and a specialization map $\mathcal{X}(\mathcal{O}_K) \rightarrow \mathcal{X}_k(k)$, where $\mathcal{X}_k \subset \mathcal{X}$ is the special fiber. If \mathcal{X} is a proper \mathcal{O}_K -scheme, then the natural map $\mathcal{X}(\mathcal{O}_K) \rightarrow X(K)$ is a bijection. As \mathcal{O}_K is Henselian, the specialization map is surjective whenever \mathcal{X} is smooth over $S := \text{Spec}(\mathcal{O}_K)$.

If \mathcal{X} is regular, then every \mathcal{O}_K -point of \mathcal{X} factors through the smooth locus of \mathcal{X} over S , see [BLR90, Chapter 3.1, Proposition 2]. Hence if a K -variety X has a regular and proper model $\mathcal{X} \rightarrow S$, then X has a K -rational point if and only if the special fiber of the smooth locus of \mathcal{X} over S is not empty. But if \mathcal{X} is not regular, then there may exist \mathcal{O}_K -points intersecting the singular locus of \mathcal{X} over S , see Example 4.6.

The existence of weak Néron models plays an important role in the study of rational points. A weak Néron model of a smooth K -variety X is a smooth and separated model \mathcal{Z} of X , such that the natural map from $\mathcal{Z}(\mathcal{O}_K)$ to $X(K)$ is a bijection. Hence if X admits a weak Néron model, then X has a K -rational point if and only if the special fiber of this weak Néron model is not empty. It is known that every smooth and proper K -variety has a weak Néron model, see [BLR90, Chapter 3.5, Theorem 2]. But in general a weak Néron model is not unique.

The smooth locus over S of a regular, proper model of a smooth, proper K -variety X is a weak Néron model of X . There is a way to obtain a weak Néron model from any proper model, the so called Néron smoothening, see [BLR90, Chapter 3], which is constructed by blowing up singular points having sections through them. But given a singular point, it is hard to decide a priori whether there is a section containing that point. Therefore the Néron smoothening does not yield a straightforward method for constructing a weak Néron model from an arbitrary singular model.

In this article we consider the following situation. Let X be a K -variety, let L/K be a Galois extension, and let X_L be the base change of X to L . Then $G := \text{Gal}(L/K)$ acts on X_L such that $X_L/G \cong X$. Consider a model \mathcal{Y} of X_L with a good G -action, i.e. an action such that every orbit is contained in an affine open subscheme of \mathcal{Y} , extending this action on X_L . Then the quotient $\mathcal{X} := \mathcal{Y}/G$ is an \mathcal{O}_K -scheme and in fact a model of X . In general \mathcal{X} will have tame quotient singularities, and there can be \mathcal{O}_K -points through the singular locus, see Example 2.7 and Example 4.6. Note that interesting models of X_L with an action as required really exist and appear naturally. For example models of X_L obtained from models of X by base change and normalization have such a G -action, and these are exactly the techniques used to construct a model with semistable reduction in the semistable reduction theorem, see [Liu02, Chapter 10, Proposition 4.6]. Moreover, we show in Theorem 2.9 that if X is a proper and smooth K -variety, then there is always a weak Néron model of X_L to which the Galois action on X_L extends. To construct such a weak Néron model, we show in particular that the Néron smoothing as constructed in [BLR90, Chapter 3] is compatible with actions of the Galois group as described above.

As the model \mathcal{X} obtained by taking the quotient is singular in general and has sections through the singular locus, neither \mathcal{X} nor its smooth locus over S will be a weak Néron model of X . But there is a way to construct a weak Néron model of X out of a weak Néron model of X_L with a G -action extending the Galois action on the generic fiber. In this context we show the following theorem.

Theorem. (Theorem 3.1) *Let L/K be a tame Galois extension, let X be a smooth K -variety, and let X_L be the base change of X to L . Let \mathcal{Y} be a smooth model of X_L with a $G := \text{Gal}(L/K)$ -action extending the Galois action on X_L . Let $\mathcal{X} := \mathcal{Y}/G$ be the quotient.*

Then there is a smooth model \mathcal{Z} of X and a separated $S := \text{Spec}(\mathcal{O}_K)$ -morphism $\Phi : \mathcal{Z} \rightarrow \mathcal{X}$, such that the induced map $\mathcal{Z}(\mathcal{O}_K) \rightarrow \mathcal{X}(\mathcal{O}_K)$ is a bijection, and such that for all smooth, integral S -schemes \mathcal{V} and all dominant S -morphisms $\Psi : \mathcal{V} \rightarrow \mathcal{X}$ there is a unique S -morphism $\Psi' : \mathcal{V} \rightarrow \mathcal{Z}$ such that $\Phi \circ \Psi' = \Psi$. In particular \mathcal{Z} is unique.

If \mathcal{Y} is a weak Néron model of X_L , then \mathcal{Z} is a weak Néron model of X .

In fact, \mathcal{Z} is the fixed locus of some G -action on the Weil restriction of \mathcal{Y} to S , see Construction 3.1. The construction goes back to [Edi92], where it is used in the context of abelian varieties and Néron models.

Note that the uniqueness of \mathcal{Z} with its properties is interesting, because in general a weak Néron model is, in contrast to a Néron model, not unique.

The theorem and its proof also yield an explicit description of a weak Néron model \mathcal{Z} of X . Having this description at hand, we can examine its special fiber \mathcal{Z}_k which is important for finding K -rational points of X . We show the following key lemma.

Lemma. (Lemma 4.1) *Let \mathcal{Y}^G be the fixed locus of the G -action on \mathcal{Y} . Then there is a k -morphism $b : \mathcal{Z}_k \rightarrow \mathcal{Y}^G$ such that for any point $y \in \mathcal{Y}^G$ with residue field $\kappa(y)$ the inverse image of y is isomorphic to $\mathbb{A}_{\kappa(y)}^m$ as $\kappa(y)$ -schemes for some $m \in \mathbb{N}$.*

To show this lemma we use the explicit description of \mathcal{Z} and of the G -action on the complete local ring of a fixed point, which is examined in Lemma 7.1 and Lemma 7.5.

There are some interesting applications of the key lemma. For example we deduce from it that the quotient \mathcal{X} has \mathcal{O}_K -points if and only if $\mathcal{Y}^G \neq \emptyset$, see Corollary 4.4. In fact these \mathcal{O}_K -point will pass through the image of \mathcal{Y}^G in \mathcal{X} , which in general will be singular. Hence we obtain examples of singular models with section through the singular locus.

We can use the obtained results also to study certain motivic invariants, the motivic Serre invariant and the rational volume. The motivic Serre invariant $S(X)$ of a K -variety X is defined to be the class of the special fiber of a weak Néron model of X in some quotient of the Grothendieck ring of varieties, namely in $K_0^{\mathcal{O}_K}(\mathrm{Var}_k)/(\mathbb{L} - 1)$, see Definition 5.2. The Serre invariant is interesting in the context of rational points, because it vanishes if X has no K -rational point. From the key lemma we deduce the following theorem.

Theorem. (Theorem 5.2) *Let X be a smooth, proper K -variety. Let L/K be a tame Galois extension, X_L the base change of X to L . Let \mathcal{Y} be a weak Néron model of X_L with a good $G := \mathrm{Gal}(L/K)$ -action extending the Galois action on X_L . Then*

$$S(X) = [\mathcal{Y}^G] \in K_0^{\mathcal{O}_K}(\mathrm{Var}_k)/(\mathbb{L} - 1).$$

The rational volume $s(X)$ of a K -variety X is defined to be the Euler characteristic with proper support and coefficients in \mathbb{Q}_l , $l \neq \mathrm{char}(k)$ a prime, of the special fiber of a weak Néron model of X . The rational volume vanishes if X has no K -rational point, too.

Theorem. (Theorem 5.4) *Let X be a smooth, proper K -variety, and let L/K be a tame Galois extension of degree q^r , q a prime. Then $s(X) = s(X_L) \pmod{q}$.*

The proof of this theorem uses the fact that there is always a weak Néron model of X_L with an action of $\mathrm{Gal}(L/K)$ extending the Galois action on X_L , see Theorem 2.9, as well as the equation for the Serre invariant (Theorem 5.2). Moreover, we use the fact that for a scheme of finite type V over some field with a good action of a q -group G , we have $\chi_c(V) = \chi_c(V^G) \pmod{q}$. This argument goes back to [Ser09, Section 7.2].

Finally, we can deduce the existence of rational points for some varieties with potential good reduction. By definition, a K -variety X has potential good reduction if there is a Galois extension L/K such that the base change of X to L admits a smooth and proper model.

Corollary. (Corollary 6.1) *Let X be a smooth, proper K -variety with potential good reduction after a base change of order q^r , $q \neq \mathrm{char}(k)$ a prime. If the Euler characteristic of X with coefficients in \mathbb{Q}_l , $l \neq \mathrm{char}(k)$ a prime, does not vanish modulo q , then X has a K -rational point.*

To prove this corollary we use Theorem 5.4, and the fact that the Euler characteristic with coefficients in \mathbb{Q}_l is constant on the fibers of a smooth and proper morphism.

In addition, we obtain a similar result for the Euler characteristic with coefficients in the structure sheaf, see Corollary 6.2. In this Corollary we need to assume that there is a tame Galois extension L/K of prime degree, such that there is a smooth and proper model of X_L with a good G -action extending the Galois action on X_L , because we cannot use the results concerning the motivic invariants. We show directly that the G -action on this smooth and proper model of X_L has a closed fixed point, and use Corollary 4.4 to conclude that in this case the model obtained by taking the quotient will have an \mathcal{O}_K -point inducing a K -point of X .

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Conventions. A variety over a field F is a geometrically integral, separated F -scheme of finite type over F . We assume that an integral scheme is connected. All schemes are assumed to be noetherian.

If U is a V -scheme, $\mathrm{Spec}(F) \rightarrow V$ any point. We set $U_F := U \times_V \mathrm{Spec}(F)$.

In the entire article, let K be a complete local field with ring of integers \mathcal{O}_K , $S := \mathrm{Spec}(\mathcal{O}_K)$, and residue field k . Assume that k is algebraically closed.

2. MODELS WITH GALOIS ACTIONS

Definition 2.1. Let X be a K -variety. A *model* of X is an integral S -scheme \mathcal{X} of finite type over S such that $\mathcal{X}_K \cong X$.

Remark 2.1. Let X be a non-empty K -variety, and let $\mathcal{X} \rightarrow S$ be any model of X . Then \mathcal{X} dominates S , so by [Har77, Chapter III, Proposition 9.7] \mathcal{X} is flat over S .

Remark 2.2. Let $\varphi : \mathcal{X} \rightarrow S$ be a model of a K -variety X . Then we have maps as follows induced by the universal property of the fiber product.

$$X(K) \longleftarrow \mathcal{X}(\mathcal{O}_K) \xrightarrow{s} \mathcal{X}_k(k)$$

If φ is proper, $\mathcal{X}(\mathcal{O}_K) \rightarrow X(K)$ is bijective by the valuative criterion of properness. If φ is smooth, the specialization map s is surjective by [BLR90, Chapter 2.3, Proposition 5], because \mathcal{O}_K is Henselian.

Definition 2.2. A *weak Néron model* of a smooth K -variety X is a smooth and separated model $\mathcal{X} \rightarrow S$ of X , such that the natural map $\mathcal{X}(\mathcal{O}_K) \rightarrow X(K)$ is a bijection.

Remark 2.3. Let X be a smooth K -variety attached with a weak Néron model $\mathcal{X} \rightarrow S$. Then $X(K) = \emptyset$ if and only if the special fiber \mathcal{X}_k of $\mathcal{X} \rightarrow S$ is empty. This is true, because by definition the natural map $\mathcal{X}(\mathcal{O}_K) \rightarrow X(K)$ is a bijection, the specializing map $\mathcal{X}(\mathcal{O}_K) \rightarrow \mathcal{X}_k(k)$ is surjective by Remark 2.2, and k is algebraically closed.

Remark 2.4. A weak Néron model does not exist for all smooth K -varieties X . It follows from [BLR90, Chapter 3.5, Theorem 2] that a weak Néron model exists if X is proper over K .

Note that a weak Néron model is not unique. Take any weak Néron model, blow up a point in the special fiber, and then take the smooth locus of the obtained scheme. This is again a weak Néron model.

Now fix a Galois extension L/K with Galois group $G := \mathrm{Gal}(L/K)$. Let \mathcal{O}_L be the ring of integers of L , $T := \mathrm{Spec}(\mathcal{O}_L)$. Note that k is the residue field of L . For a general introduction to local fields and their Galois extensions we refer to [Ser79]. A Galois extension L/K is called *tame*, if the order of its Galois group is prime to $\mathrm{char}(k)$. From [Ser79, Chapter IV, Corollary 2 and Corollary 4] we get that the Galois group of a tame Galois extension L/K is always cyclic.

We now want to consider group actions of the Galois group. Therefore, recall the following facts concerning group actions of an abstract finite group G .

Let U be a scheme, $\mathrm{Aut}(U)$ the abstract group of automorphisms of U . A G -action on U is given by a group homomorphism $\mu_U : G \rightarrow \mathrm{Aut}(U)$. If U is an affine scheme,

i.e. $U = \text{Spec}(A)$, then a group action on U is also given by a group homomorphism $\mu_U^\# : G \rightarrow \text{Aut}(A)$.

Let U, V be schemes with G -actions. We call a morphism of schemes $f : U \rightarrow V$ *G-equivariant*, if for all $g \in G$ we have $f \circ \mu_U(g) = \mu_V(g) \circ f$.

A G -action on a scheme U is called *good*, if every orbit is contained in an affine open subscheme of U . By [Gro63, Exposé V, Proposition 1.8] this is the same as requiring a cover of U by affine, open, G -invariant subschemes.

If U is a scheme with a good G -action, then there exists a quotient $\pi : U \rightarrow U/G$ in the category of schemes, see [Gro63, Exposé V.1].

Remark 2.5. By definition of the Galois group, G acts on L , and $K = L^G$. The G -action of L can be restricted to \mathcal{O}_L , and $\mathcal{O}_L^G = \mathcal{O}_K$. We call this action the *Galois action* on \mathcal{O}_L . Note that $\text{Spec}(L) \hookrightarrow T$ is G -equivariant for these actions. Let X be a K -variety. As X is flat over K , by [Gro63, Exposé V, Proposition 1.9], G acts on X_L such that $X_L \rightarrow \text{Spec}(L)$ is G -invariant and $X_L/G \cong X$. We call this action the *Galois action* on X_L .

Remark 2.6. Let X be a K -variety. Let $\varphi : \mathcal{Y} \rightarrow T$ be a model of X_L with a good G -action. Assume that $X_L \hookrightarrow \mathcal{Y}$ is G -equivariant for the action on \mathcal{Y} and the Galois action on X_L , i.e. the G -action on \mathcal{Y} extends the Galois action on X_L .

Take any $h \in G$, and let $g \in \text{Aut}(\mathcal{Y})$ and $g_T \in \text{Aut}(T)$ be its images. As the maps $X_L \hookrightarrow \mathcal{Y}$, $X_L \rightarrow \text{Spec}(L)$, and $\text{Spec}(L) \hookrightarrow T$ are $\text{Gal}(L/K)$ -equivariant, we obtain that $g_T \circ \varphi \circ g^{-1}|_{X_L} = \varphi|_{X_L}$. As $X_L \subset \mathcal{Y}$ is open and dense, \mathcal{Y} is reduced, and T is separated, [GW10, Corollary 9.9] implies that $g_T \circ \varphi \circ g^{-1} = \varphi$, i.e. φ is G -equivariant.

Let $\pi : \mathcal{Y} \rightarrow \mathcal{X} := \mathcal{Y}/G$ be the quotient. Using that the maps in the square on the left hand side are G -equivariant, we get the following big commutative diagram.

$$\begin{array}{ccccc}
 & & \pi & & \\
 X_L & \xrightarrow{\quad} & \mathcal{Y} & \xrightarrow{\quad} & X & \xrightarrow{\quad} & \mathcal{X} \\
 \downarrow & \square & \downarrow \varphi & & \downarrow & & \downarrow \\
 \text{Spec}(L) & \xrightarrow{\quad} & T & \xrightarrow{\quad} & \text{Spec}(K) & \xrightarrow{\quad} & S
 \end{array}$$

Note that \mathcal{X} is an S -scheme of finite type by [Gro63, Exposé V, Proposition 1.5]. As \mathcal{X} is a quotient by a finite group of the integral scheme \mathcal{Y} , it is integral, too. As $\text{Spec}(L) \hookrightarrow T$ is flat, by [Gro63, Exposé V, Proposition 1.9] we obtain

$$\mathcal{X}_K = \mathcal{Y}/G \times_S \text{Spec}(K) \cong \mathcal{Y} \times_S \text{Spec}(K)/G = X_L/G = X.$$

Hence $\mathcal{X} \rightarrow S$ is a model of X .

In general the quotient \mathcal{X} will be singular. To see this, look at the following example.

Example 2.7. Let k be an algebraically closed field with $\text{char}(k) \neq 2$, and set $K := k((s))$, $L := k((t))$ with $t^2 = s$. Hence L/K is a tame Galois extension with Galois group $G = \mathbb{Z}/2\mathbb{Z}$. The Galois action on $k((t))$ is given by

$$\alpha : k((t)) \rightarrow k((t)); P(t) \mapsto P(-t).$$

Set $X := \mathbb{A}_K^1 = \text{Spec}(k((s))[y])$. The Galois action on $X_L = \text{Spec}(k((t))[y])$ is given by

$$\beta : k((t))[y] \rightarrow k((t))[y]; P(t, y) \mapsto P(-t, y).$$

Look at the smooth $\mathcal{O}_L = k[[t]]$ -scheme $\mathcal{Y} := \mathbb{A}_{k[[t]]}^1 = \text{Spec}(k[[t]][x])$ with the G -action given by

$$\gamma : k[[t]][x] \rightarrow k[[t]][x]; P(t, x) \mapsto P(-t, -x).$$

Using the fact that t is invertible in $k((t))$, one shows that the map

$$X_L = \operatorname{Spec}(k((t))(y)) \rightarrow \mathcal{Y}_L = \operatorname{Spec}(k((t))(x))$$

given by sending t to t and y to xt is a G -equivariant isomorphism. Hence \mathcal{Y} is a model of X_L with a G -action extending the Galois action on X_L .

The $k[[t]]^G = k[[t^2]] = k[[s]] = \mathcal{O}_K$ -scheme

$$\mathcal{X} := \mathcal{Y}/G = \operatorname{Spec}(k[[t]][x]^G) = \operatorname{Spec}(k[[t^2]][tx, x^2]) \cong \operatorname{Spec}(k[[s]][b, c]/(sc - b^2))$$

is singular in $(0, 0, 0)$, so by Remark 2.6 it \mathcal{X} is a singular model of X .

Remark 2.8. In order to get a projective example, replace in Example 2.7 $X = \mathbb{A}_K^1$ by \mathbb{P}_K^1 and $\mathcal{Y} = \mathbb{A}_{k[[t]]}^1$ by $\mathbb{P}_{k[[t]]}^1$ with a G -action given by $g \in \operatorname{Aut}(\mathbb{P}_{k[[t]]}^1)$ such that

$$g(t, [x_0 : x_1]) = (-t, [-x_0 : x_1]).$$

For a given K -variety X and a Galois extension L/K , there exist interesting models of X_L with a good action of the Galois group as in Remark 2.6. In this context we can show the following theorem.

Theorem 2.9. *Let L/K be a Galois extension with Galois group $G := \operatorname{Gal}(L/K)$, and let \mathcal{O}_L be the ring of integers of L , $T := \operatorname{Spec}(\mathcal{O}_L)$. Then for a given smooth and proper K -variety X , there exists a weak Néron model $\varphi : \mathcal{Y} \rightarrow T$ of X_L and a good G -action on \mathcal{Y} extending the Galois action on X_L .*

Proof. In order to prove this theorem, we need to recall how a weak Néron model is constructed. The main tool of showing that weak Néron models actually exist is the so called Néron smoothening.

Definition 2.3. Let X be a smooth K -variety, and let $\mathcal{X} \rightarrow S$ be a model of X . A *Néron smoothening* of \mathcal{X} is a proper S -morphism $f : \mathcal{X}' \rightarrow \mathcal{X}$ such that f is an isomorphism on the generic fibers, and such that the canonical map $\operatorname{Sm}(\mathcal{X}'/S)(S) \rightarrow \mathcal{X}'(S)$ is bijective. Here $\operatorname{Sm}(\mathcal{X}'/S)$ is the smooth locus of \mathcal{X}' over S .

In order to prove Theorem 2.9 we need the following Lemma.

Lemma 2.10. *Let Y be a smooth L -variety, let $\mathcal{Y} \rightarrow T$ be a model of Y with a good G -action, and assume that the structure map $\varphi : \mathcal{Y} \rightarrow T$ is G -equivariant for this action and the Galois action on T . Then there exists a projective Néron smoothening $f : \mathcal{Y}' \rightarrow \mathcal{Y}$, and a good G -action on \mathcal{Y}' such that f is G -equivariant.*

Proof. By [BLR90, Chapter 3.1, Theorem 3] there exists a Néron smoothening $f : \mathcal{Y}' \rightarrow \mathcal{Y}$, which consists of a finite sequence of blowups with centers in the special fibers. We need to construct a G -action on \mathcal{Y}' such that f is G -equivariant. Note that if we blow up an integral scheme U with a good G -action in a closed G -invariant subscheme $V \subset U$, and denote by $u : U' \rightarrow U$ the blowup, then there is a G -action on U' such that u is G -equivariant. The reason for this is the following. Let $h \in G$ be any element, $g_U \in \operatorname{Aut}(U)$ its image. As V is G -invariant, $g_U(V) = V$. So by the universal property of blowup, see [Har77, Chapter II, Corollary 7.15], there exists a unique $g_{U'} \in \operatorname{Aut}(U')$ such that $u \circ g_{U'} = g_U \circ u$. This way we can define the required group action on U' , and u is G -equivariant by construction.

Consider f , which is a sequence of blowups, i.e. we have

$$\mathcal{Y}' =: \mathcal{Y}_m \xrightarrow{f_{m-1}} \mathcal{Y}_{m-1} \xrightarrow{f_{m-2}} \dots \xrightarrow{f_1} \mathcal{Y}_1 \xrightarrow{f_0} \mathcal{Y}_0 := \mathcal{Y}.$$

$\underbrace{\hspace{15em}}_f$

Here the f_i are blowups of some closed subschemes $V_i \subset \mathcal{Y}_i$. One checks in the proof of [BLR90, Chapter 3.4, Theorem 2] that all the V_i are obtained using the

same construction. Hence if we show that $V := V_0 \subset \mathcal{Y}$ is G -invariant, then we obtain a G -action on \mathcal{Y}_1 such that f_0 is G -equivariant, hence $\varphi \circ f_0$ is G -equivariant, and we can conclude inductively on the length of the sequence of blowups.

One can check in [BLR90, Chapter 3.4, Theorem 2] that V is constructed as follows. Let $E \subset \mathcal{Y}(\mathcal{O}_L)$ be the subset of all $\sigma \in \mathcal{Y}(\mathcal{O}_L)$ not factoring through $\text{Sm}(\mathcal{Y}/T)$, with $\text{Sm}(\mathcal{Y}/T)$ the smooth locus of \mathcal{Y} over T . Set $F^1 := E$. Let $s : \mathcal{Y}(\mathcal{O}_L) \rightarrow \mathcal{Y}_k(k)$ be the specialization map, and V^i the Zariski closure of $s(F^i)$ in \mathcal{Y} , and let $U^i \subset V^i$ be the largest open subset, such that U^i is smooth over k , and that $\Omega_{\mathcal{Y}/T}^1|_{V^i}$ is locally free over U^i . Set $E^i := \{a \in F^i \mid s(a) \in U^i\}$, and $F^{i+1} := F^i \setminus E^i$. Note that there is a minimal $t \in \mathbb{N}$ such that $F^{t+1} = \emptyset$. Set $V = V^t$.

The action of G on \mathcal{Y} induces a G -action on $\mathcal{Y}(\mathcal{O}_L)$, and, as φ is G -equivariant, and hence $\mathcal{Y}_k \subset \mathcal{Y}$ is G -invariant, a G -action on $\mathcal{Y}_k(k)$. Note that s is G -equivariant. We now show by induction that F^i is G -invariant for all i .

Take any $h \in G$, let g be its image in $\text{Aut}(\mathcal{Y})$, g_T its image in $\text{Aut}(T)$. Note that

$$\varphi|_{g(\text{Sm}(\mathcal{Y}/T))} = g_T \circ \varphi \circ g^{-1}|_{g(\text{Sm}(\mathcal{Y}/T))} = g_T \circ \varphi|_{\text{Sm}(\mathcal{Y}/T)} \circ g^{-1}|_{g(\text{Sm}(\mathcal{Y}/T))}.$$

Hence $\varphi|_{g(\text{Sm}(\mathcal{Y}/T))}$ is smooth, which implies that $\text{Sm}(\mathcal{Y}/T)$ is G -invariant, hence $E = F^1$ is G -invariant. So we may assume that F^i is G -invariant for some i .

Consider $Z_i := \cap_{h \in G} h(V^i) \subset V^i$. By construction, Z_i is closed in \mathcal{Y} and G -invariant. As F^i is G -invariant by assumption, $s(F^i) \subset V^i$ is G -invariant, and hence $s(F^i) \subset h(V^i)$ for all $h \in G$, i.e. $s(F^i) \subset Z_i$. So by definition of the Zariski closure, $V^i = Z_i$, hence in particular V^i is G -invariant.

Let $\text{Sm}(V^i)$ be the smooth locus of V^i over k . Note that $U^i = \text{Sm}(V^i) \cap W^i$, with $W^i \subset V^i$ the largest open subset over which $\Omega_{\mathcal{Y}/T}^1|_{V^i}$ is locally free. As the G -action on V^i is given by isomorphisms, regular points are mapped to regular points, hence $\text{Sm}(V^i)$ is G -invariant. Now we examine W^i . Since G acts equivariantly on $\mathcal{Y} \rightarrow T$, the natural map $g^*(\Omega_{\mathcal{Y}/T}^1) \rightarrow \Omega_{\mathcal{Y}/T}^1$ is an isomorphism. As V^i is G -invariant, $g^*(\Omega_{\mathcal{Y}/T}^1)|_{V^i} \rightarrow \Omega_{\mathcal{Y}/T}^1|_{V^i}$ is an isomorphism, too. Altogether we obtain

$$\Omega_{\mathcal{Y}/T}^1|_{V^i \cap W^i} \cong g^*(\Omega_{\mathcal{Y}/T}^1)|_{V^i \cap W^i} = g^*(\Omega_{\mathcal{Y}/T}^1|_{V^i \cap g^{-1}(W^i)}).$$

As the first is locally free by definition of W^i , $g^*(\Omega_{\mathcal{Y}/T}^1|_{V^i \cap g^{-1}(W^i)})$ is locally free, too. As g is an automorphism of \mathcal{Y} , $\Omega_{\mathcal{Y}/T}^1|_{V^i \cap g^{-1}(W^i)}$ is locally free. Hence by definition of W^i , $g^{-1}(W^i) \subset W^i$, i.e. W^i is G -invariant. Hence $U^i = \text{Sm}(V^i) \cap W^i$ is G -invariant, too.

Using that s is G -equivariant, we get that E^i is G -invariant, and therefore F^{i+1} is G -invariant, too. So it follows by induction that for all i , F^i is G -invariant, in particular F^t is G -invariant. Using the same argument as in the induction, we can show that $V^t = V$ is G -invariant, and this is what we wanted to show.

We still need to show that the G -action on \mathcal{Y} is good. So take any orbit in \mathcal{Y}' . Its image under f will be contained in an open affine subset $U \subset \mathcal{Y}$. As f is projective, $f^{-1}(U)$ is projective over U and contains our orbit, which is finite, because G is finite. By [Liu02, Chapter 3, Proposition 3.36.b] there is an affine subset $U' \subset f^{-1}(U)$ containing every finite set of points. Hence the action on \mathcal{Y}' is good. \square

In [Nic12] the following similar theorem in the context of formal schemes is proven:

Theorem. *Any generically smooth, flat, separated formal \mathcal{O}_L -scheme X_∞ , topologically of finite type over \mathcal{O}_L , endowed with a good G -action compatible with the G -action on \mathcal{O}_L , admits a G -equivariant Néron smoothening.*

Now we are finally ready to prove Theorem 2.9. So let X be a smooth, proper K -variety. In particular X is a separated S -scheme of finite type, hence by Nagata's

embedding Theorem, see [GW10, Theorem 12.70], there exists a proper, integral S -scheme \mathcal{X} and an immersion $X \hookrightarrow \mathcal{X}$ over S which is schematically dense. As X is proper over K , \mathcal{X}_K is in fact isomorphic to X . Altogether, $\mathcal{X} \rightarrow S$ is a proper model of X .

Set $\mathcal{X}_T := \mathcal{X} \times_S T$, and let $\Phi : \mathcal{X}_T \rightarrow T$ be the projection to T . Note that Φ is proper, and $\mathcal{X}_T \times_T \text{Spec}(L) = X_L$. By Remark 2.1, \mathcal{X} is flat over S , therefore \mathcal{X}_T is flat over T . Hence there cannot be a connected component of \mathcal{X}_T only supported on the special fiber. But the generic fiber X_L of \mathcal{X}_T is connected, hence \mathcal{X}_T is connected. Hence one can check locally that \mathcal{X}_T is integral, which is straightforward to check. Altogether, $\Phi : \mathcal{X}_T \rightarrow T$ is a proper model of X .

As $\mathcal{X} \rightarrow S$ is flat, by [Gro63, Exposé V, Proposition 1.9] there exists a good G -action on \mathcal{X}_T such that Φ is G -equivariant, and $\mathcal{X}_T/G \cong \mathcal{X}$. This G -action extends the Galois action on X_L by construction.

By Lemma 2.10 there exists a projective Néron smoothening $f : \mathcal{Y}' \rightarrow \mathcal{X}_T$, and a good G -action on \mathcal{Y}' such that f is G -equivariant. Let $\mathcal{Y} \subset \mathcal{Y}'$ be the smooth locus of $\Phi \circ f$. Set $\varphi := \Phi \circ f|_{\mathcal{Y}}$. Note that φ is separated. We have the following commutative diagram.

$$\begin{array}{ccc} \mathcal{Y}' & \xrightarrow{\quad} & \mathcal{Y}' \\ & \searrow \varphi & \downarrow f \\ & & \mathcal{X}_T \\ & & \downarrow \Phi \\ & & T \end{array}$$

Note that X_L is smooth, because X is smooth. As f is a Néron smoothening, $\mathcal{Y}'_L = \mathcal{X}_T \times_T \text{Spec}(L) = X_L$. Hence \mathcal{Y}'_L is in particular smooth over T , so $\mathcal{Y}_L = \mathcal{Y}'_L = X_L$. As \mathcal{X}_T is integral, \mathcal{Y}' and \mathcal{Y} are integral, too. Hence $\varphi : \mathcal{Y} \rightarrow T$ is a smooth and separated model of X_L . As Φ and f are proper, by the valuative criterion of properness the natural map $\mathcal{Y}'(\mathcal{O}_L) \rightarrow X_L(L)$ is a bijection. As f is a Néron smoothening, $\mathcal{Y}'(\mathcal{O}_L) = \mathcal{Y}(\mathcal{O}_L)$. So $\varphi : \mathcal{Y} \rightarrow T$ is a weak Néron model of X_L . We still need to show that there is a good G -action on \mathcal{Y} extending the Galois action on X_L . As f is G -invariant for the G -action on \mathcal{Y}' and \mathcal{X}_T , the G -action \mathcal{Y}' extends the Galois action on X_L . So it suffices to show that this G -action restricts to \mathcal{Y} , i.e. that $\mathcal{Y} \subset \mathcal{Y}'$ is G -invariant. To show this, we can simply use the proof in the base case of the induction in Lemma 2.10. Note that the action is good for the following reason. Take any orbit in \mathcal{Y} . As the action on \mathcal{Y}' is good, it is contained in an open affine subset $U \subset \mathcal{Y}'$. So it is contained in $U \cap \mathcal{Y}$, which is open in U . So by [Liu02, Chapter 3, Proposition 3.36.b] there is an affine subset $U' \subset U \cap \mathcal{Y}$ containing our finite orbit. \square

In [EN11, Proposition 4.5] the following similar statement is proven.

Proposition. *Let G be any finite group, X a smooth and proper K -variety, endowed with a good G -action. Then there is a weak Néron model $\mathcal{X} \rightarrow S$ of X endowed with a good G -action, such that $X \hookrightarrow \mathcal{X}$ is G -equivariant.*

3. A CANONICAL WEAK NÉRON MODEL OF A QUOTIENT SCHEME

Theorem 3.1. *Let L/K be a tame Galois extension, $G := \text{Gal}(L/K)$. Let \mathcal{O}_L be the ring of integers of L , $T := \text{Spec}(\mathcal{O}_L)$. Let X be a smooth K -variety, and let $\varphi : \mathcal{Y} \rightarrow T$ be a smooth model of X_L with a good G -action extending the Galois action on X_L . Let $\mathcal{X} := \mathcal{Y}/G$ be the quotient.*

Then there is a smooth model $\mathcal{Z} \rightarrow S$ of X and a separated S -morphism $\Phi : \mathcal{Z} \rightarrow \mathcal{X}$, such that the induced map $\mathcal{Z}(S) \rightarrow \mathcal{X}(S)$ is a bijection, and such that for all smooth

integral S -schemes \mathcal{V} and all dominant S -morphisms $\Psi : \mathcal{V} \rightarrow \mathcal{X}$ there is a unique S -morphism $\Psi' : \mathcal{V} \rightarrow \mathcal{Z}$ making the following diagram commutative.

$$\begin{array}{ccc} \mathcal{V} & & \\ \downarrow \Psi & \searrow \Psi' & \\ & \mathcal{Z} & \\ & \swarrow \Phi & \\ & \mathcal{X} & \end{array}$$

In particular \mathcal{Z} is unique with this property up to a unique isomorphism.

If $\mathcal{Y} \rightarrow T$ is a weak Néron model of X_L , then $\mathcal{Z} \rightarrow S$ is a weak Néron model of X .

Proof. The proof consists of six steps. First we will give the construction of \mathcal{Z} as a functor of schemes, then we construct Φ as a morphism of functors. In the third step we will show that \mathcal{Z} is represented by a smooth S -scheme. Thereafter we show the properties of Φ , namely that it is separated and that the map $\mathcal{Z}(S) \rightarrow \mathcal{X}(S)$ induced by Φ is an isomorphism. Afterwards we show the universal property. In the final step we consider the case that \mathcal{Y} is a weak Néron model of X_L .

Construction of \mathcal{Z} . We now construct \mathcal{Z} . The construction can be found in [Edi92], where it is used in the context of abelian varieties.

Definition 3.1. The *Weil restriction* of a T -scheme U to S is defined as the functor

$$\begin{aligned} \text{Res}_{T/S}(U) : (\text{Sch}/S) &\rightarrow (\text{Sets}) \\ W &\mapsto \text{Hom}_T(W \times_S T, U). \end{aligned}$$

Definition 3.2. Let V be an S -scheme with a G -action, such that the structure map is G -equivariant for this action and the trivial action on S . We define the *functor of fixed points* by

$$\begin{aligned} V^G : (\text{Sch}/S) &\rightarrow (\text{Sets}) \\ W &\mapsto V(W)^G = \text{Hom}_S(W, V)^G. \end{aligned}$$

By [Edi92, Proposition 3.1] this functor is represented by a subscheme of V . We call this scheme the *fixed locus* of the G -action on V .

Note that G is a finite cyclic group, because L/K is a tame Galois extension. Therefore every G -action is given by one automorphism.

Construction 3.1. [Edi92, Construction 2.4 and Theorem 4.2]

Fix a generator of G , and let $g \in \text{Aut}(\mathcal{Y})$ and $g_T \in \text{Aut}(T)$ be its images. Then $\tilde{g} \in \text{Aut}(\text{Res}_{T/S}(\mathcal{Y}))$, which maps $f \in \text{Hom}_T(W \times_S T, \mathcal{Y})$ to $g \circ f \circ (\text{id}_W \times g_T)^{-1}$ for every $W \in (\text{Sch}/S)$, defines a G -action on $\text{Res}_{T/S}(\mathcal{Y})$. It is easy to see that \tilde{g} is an S -morphism. Therefore the structure map $\text{Res}_{T/S}(\mathcal{Y}) \rightarrow S$ is G -equivariant for the G -action on $\text{Res}_{T/S}(\mathcal{Y})$ and the trivial G -action on S . Define

$$\begin{aligned} \mathcal{Z} : (\text{Sch}/S) &\rightarrow (\text{Sets}) \\ W &\mapsto (\text{Res}_{T/S}(\mathcal{Y}))^G(W) = \text{Hom}_T(W \times_S T, \mathcal{Y})^G. \end{aligned}$$

Note that $\text{Hom}_T(W \times_S T, \mathcal{Y})^G$ is the set of G -equivariant T -morphisms from $W \times_S T$ to \mathcal{Y} .

Construction of Φ . View \mathcal{X} and \mathcal{Z} as functors from the category of flat S -schemes to the category of sets. We now construct a morphism of functors $\Phi : \mathcal{Z} \rightarrow \mathcal{X}$. As soon as we will have shown that \mathcal{Z} is in fact representable by a flat S -scheme, this will yield an S -morphism of schemes by Yoneda's lemma for the category of flat S -schemes.

We need to construct maps $\Phi(W) : \mathcal{Z}(W) = \text{Hom}_T(W \times_S T, \mathcal{Y})^G \rightarrow \mathcal{X}(W)$ for all flat $W \in (\text{Sch}/S)$, and show that they are functorial. Take any $f \in \mathcal{Z}(W)$. Let $\pi : \mathcal{Y} \rightarrow \mathcal{X}$ be the quotient map. We have the following commutative diagram.

$$\begin{array}{ccccc}
 & & \mathcal{Y} & & \\
 & \nearrow f & \downarrow \varphi & \searrow \pi & \\
 W \times_S T & \longrightarrow & T & \longrightarrow & \mathcal{X} \\
 \downarrow p_W & \searrow f' & \downarrow & \nearrow & \\
 W & \longrightarrow & S & &
 \end{array}$$

As $W \rightarrow S$ is flat, [Gro63, Exposé V, Proposition 1.9] implies that the projection map $p_W : W \times_S T \rightarrow W$ is the quotient of the G -action on $W \times_S T$ given by $\text{id}_W \times g_T$. As $f = g \circ f \circ (\text{id}_W \times g_T)^{-1}$, and π is G -equivariant for the G -action on \mathcal{Y} and the trivial action on \mathcal{X} , we get $(\pi \circ f) \circ (\text{id}_W \times g_T) = \pi \circ g \circ f = \pi \circ f$. Hence $\pi \circ f$ is G -equivariant for the G -action on $W \times_S T$ and the trivial action on \mathcal{X} , and therefore, by the universal property of the quotient $p_W : W \times_S T \rightarrow W$ we obtain a unique $f' \in \mathcal{X}(W)$ making the diagram above commutative. We set $\Phi(W)(f) := f'$. It is easy to check that this map is functorial.

View $\mathcal{X}_K \cong X$ as a presheaf on the category of K -schemes. We now construct an inverse map of functors $\Phi|_{\mathcal{Z}_K}^{-1} : X \rightarrow \mathcal{Z}_K \subset \mathcal{Z}$. Take any $W \in (\text{Sch}/K)$. Note that $W \times_S T \cong W_L$, hence

$$\mathcal{Z}_K(W) = \text{Hom}_T(W \times_S T, \mathcal{Y})^G = \text{Hom}_T(W_L, \mathcal{Y})^G = \text{Hom}_L(W_L, X_L)^G.$$

Take any $h \in X(W)$, and consider the following diagram with p_L and p_W the projection maps.

$$\begin{array}{ccccc}
 W_L & \xrightarrow{h \circ p_W} & X & & \\
 \downarrow p_L & \searrow h^* & \downarrow \varphi|_{X_L} & \searrow \pi|_{X_L} & \\
 & X_L & \longrightarrow & X & \\
 & \downarrow \varphi|_{X_L} & \square & \downarrow & \\
 & \text{Spec}(L) & \longrightarrow & \text{Spec}(K) &
 \end{array}$$

As h is a K -morphism, the diagram commutes, and hence the universal property of fiber product induces a unique $h^* \in \text{Hom}_L(W_L, X_L)$ with $\pi \circ h^* = h \circ p_W$. Using that h^* is unique, one can easily show that it is actually G -equivariant, hence we may set $\Phi|_{\mathcal{Z}_K}^{-1}(W)(h) := h^*$. It is straightforward to check functoriality and the fact that $\Phi|_{\mathcal{Z}_K}^{-1} \circ \Phi|_{\mathcal{Z}_K} = \text{id}_{\mathcal{Z}_K}$, and $\Phi \circ \Phi|_{\mathcal{Z}_K}^{-1} = \text{id}_X$.

Representability of \mathcal{Z} . Now we are ready to show that \mathcal{Z} is actually represented by an S -scheme. Unfortunately we cannot show that $\text{Res}_{T/S}(\mathcal{Y})$ is representable using [BLR90, Chapter 7.6, Theorem 4], because as \mathcal{Y} does not need to be quasi-projective, we cannot show that every finite set of points in \mathcal{Y} is contained in an affine subset of \mathcal{Y} . Therefore we show directly that \mathcal{Z} is representable by using methods from the proof of [BLR90, Chapter 7.6, Theorem 4].

Note that if $U \subset \mathcal{Y}$ is open and G -invariant, then $\text{Res}_{T/S}(U)^G$ is well defined, and moreover there is a natural map of functors $\text{Res}_{T/S}(U)^G \rightarrow \text{Res}_{T/S}(\mathcal{Y})^G$, because $\text{Hom}_T(W \times_S T, U)^G(W)$ is a subset of $\text{Hom}_T(W \times_S T, \mathcal{Y})^G(W)$. By [BLR90, Chapter 7.6, Proposition 2] the morphism of functors $\text{Res}_{T/S}(U) \rightarrow \text{Res}_{T/S}(\mathcal{Y})$ is an open immersion. As

$$\text{Res}_{T/S}(U)^G \cong \text{Res}_{T/S}(\mathcal{Y})^G \times_{\text{Res}_{T/S}(\mathcal{Y})} \text{Res}_{T/S}(U),$$

and as open immersion are stable under base change, $\text{Res}_{T/S}(U)^G \rightarrow \text{Res}_{T/S}(\mathcal{Y})^G$ is an open immersion.

Let $\cup U_i = \mathcal{Y}$ be a cover of \mathcal{Y} by affine, G -invariant open subsets, which exists, because the G -action on \mathcal{Y} is good. Consider any U_i . As $T \rightarrow S$ is finite and flat, and U_i is affine, by [BLR90, Chapter 7.6, Theorem 4], $\text{Res}_{T/S}(U_i)$ is represented by a scheme, and by [Edi92, Proposition 3.1], $(\text{Res}_{T/S}(U_i))^G$ is represented by a subscheme of $\text{Res}_{T/S}(U_i)$. As $\text{Res}_{T/S}(\mathcal{Y}_L)^G \cong X$ as functors, $\text{Res}_{T/S}(\mathcal{Y}_L)^G$ is representable by the scheme X .

If $U, V \subset \mathcal{Y}$ are two open, G -invariant subsets, then $U \cap V \subset \mathcal{Y}$ is also an open, G -invariant subset, and the open immersions $\text{Res}_{T/S}(U \cap V)^G \rightarrow \text{Res}_{T/S}(U)^G$ and $\text{Res}_{T/S}(U \cap V)^G \rightarrow \text{Res}_{T/S}(V)^G$ define a gluing data for $\text{Res}_{T/S}(U)^G$ and $\text{Res}_{T/S}(V)^G$. One computes that in fact

$$(1) \quad \text{Res}_{T/S}(U)^G \times_{\mathcal{Z}} \text{Res}_{T/S}(V)^G = \text{Res}_{T/S}(U \cap V)^G.$$

Let \mathcal{S} be the scheme constructed by gluing X and the $\text{Res}_{T/S}(U_i)^G$ as explained above. We get a map $\iota : \mathcal{S} \rightarrow \mathcal{Z}$, because the gluing data is compatible with the open immersions $X \rightarrow \mathcal{Z}$ and $\text{Res}_{T/S}(U_i)^G \rightarrow \mathcal{Z}$. As $\iota|_X$ and the $\iota|_{\text{Res}_{T/S}(U_i)^G}$ are open immersions, and equation (1) holds pairwise for X and the $\text{Res}_{T/S}(U_i)^G$, ι is an open immersion.

We now want to show that ι is an equivalence of functors. Considering the last paragraph of the proof of [BLR90, Chapter 7.6, Theorem 4] it suffices to show the following. For every field F with a map $\text{Spec}(F) \rightarrow S$ every T -morphism $f : \text{Spec}(F) \times_S T \rightarrow \mathcal{Y}$ factors either through X_L or through one of the U_i . If F lies over K , f will factor through X_L . If F lies over k , $f(\text{Spec}(F) \times_S T)$ will only be a point topologically, so f will factor through every open neighborhood of that point. As the U_i cover \mathcal{Y} , there is a U_i through which f factors.

Altogether ι is an equivalence of functors, and therefore \mathcal{Z} is represented by the S -scheme \mathcal{S} .

Now we show that \mathcal{Z} is a smooth S -scheme. Note that it suffices to check smoothness locally. By construction of the scheme representing \mathcal{Z} , every point in \mathcal{Z} lies either in $X \cong \mathcal{Z}_K$, or in $\text{Res}_{T/S}(U_i)^G$ for some i . By assumption X is smooth over S . Moreover \mathcal{Y} is smooth over T , i.e. in particular the $U_i \subset \mathcal{Y}$ are smooth over T . Hence by [BLR90, Chapter 7.6, Proposition 3], the $\text{Res}_{T/S}(U_i)$ are smooth over S . So by [Edi92, Proposition 3.4] the $\text{Res}_{T/S}(U_i)^G$ are smooth over S . Altogether \mathcal{Z} is smooth over S .

We have seen that $X \cong \mathcal{Z}_K$, and X is integral by assumption. As \mathcal{Z} is smooth over S , it is reduced, and flat over S , so there is no irreducible component only supported on the special fiber. Altogether \mathcal{Z} is integral. This yields that $\mathcal{Z} \rightarrow S$ is a smooth model of X .

Properties of Φ . In order to show that Φ is separated, take any valuation ring R with quotient field Q , and any two morphisms $f_1, f_2 \in \text{Hom}(\text{Spec}(R), \mathcal{Z})$ such that $f_1|_{\text{Spec}(Q)} = f_2|_{\text{Spec}(Q)}$, and $\Phi \circ f_1 = \Phi \circ f_2$. Let $x \in \mathcal{X}$ be the image of the closed point of R . As R is a valuation ring, $\Phi \circ f_1 = \Phi \circ f_2$ will factor through every open neighborhood of x , so we may assume that it factors through $\text{Spec}(A_i^G) \subset \mathcal{X}$ for some G -equivariant affine subset $U_i = \text{Spec}(A_i) \subset \mathcal{Y}$. Hence the f_i factor through $\Phi^{-1}(\text{Spec}(A_i^G)) \cong \text{Res}_{S/T}(U_i)^G$. By [BLR90, Chapter 7.6, Proposition 5] and [Edi92, Proposition 3.1], $\text{Res}_{S/T}(U_i)^G$ is separated over S , hence $f_1 = f_2$. So by the valuative criterion of separateness, Φ is separated.

Now look at $\mathcal{Z}(S) = \text{Hom}_T(T, \mathcal{Y})^G = \{\sigma \in \text{Hom}_T(T, \mathcal{Y}) \mid g \circ \sigma \circ g_T^{-1} = \sigma\}$. Let $\pi_T : T \rightarrow S$ be the quotient map induced by the Galois action on T . Note that $\Phi(S) : \mathcal{Z}(S) \rightarrow \mathcal{X}(S)$ maps $\sigma \in \mathcal{Z}(S)$ to $\sigma_G \in \mathcal{X}(S)$ with $\sigma_G \circ \pi_T = \pi \circ \sigma$.

Take any $\sigma_G \in \mathcal{X}(S)$. Let $\sigma'_G : \mathcal{Y} \times_{\mathcal{X}} S \rightarrow \mathcal{X}$ be the pullback of σ_G , $\pi' : \pi^{-1}(S) \rightarrow S$ the pullback of π , and set $\varphi' := \varphi \circ \sigma'_G$. By the universal property of the fiber product, we have a one to one correspondence of sections σ of φ with $\pi \circ \sigma = \sigma_G \circ \pi_T$, and sections σ' of φ' with $\pi' \circ \sigma' = \pi_T$. Note that $\pi_T \circ \varphi' = \pi'$, π_T is separated, and π' is proper, because π is a quotient map and hence proper. So φ' is proper by [GW10, Proposition 12.58]. Hence without loss of generality we may assume that φ is proper.

By assumption, $\mathcal{Y}_L \cong \mathcal{X} \times_S \text{Spec}(L)$, hence σ_G induces a unique section σ' of $\varphi|_{\mathcal{Y}_L}$ with $\pi \circ \sigma' = \sigma_G \circ \pi_T|_{\text{Spec}(L)}$. As φ is proper, we get a unique section σ of φ with $\sigma|_{\text{Spec}(L)} = \sigma'$. As S is separated, $\pi \circ \sigma = \sigma_G \circ \pi_T$. We still need to show that $\sigma \in \mathcal{Z}(S)$, i.e. that $\sigma = g \circ \sigma \circ g_T^{-1}$. Therefore one shows that $g \circ \sigma \circ g_T^{-1}$ is a section of φ and $\pi \circ g \circ \sigma \circ g_T^{-1} = \sigma_G \circ \pi_T$, and one concludes using the uniqueness of σ with this properties. Hence σ is the unique element in $\mathcal{Z}(S)$ with $\Phi(S)(\sigma) = \sigma_G$, i.e. $\Phi(S)$ is bijective.

Universal property. Now let \mathcal{V} be a smooth, integral S -scheme and let $\Psi : \mathcal{V} \rightarrow \mathcal{X}$ be a dominant S -morphism. Assume that there exists a $\Psi' : \mathcal{V} \rightarrow \mathcal{Z}$ such that $\Phi \circ \Psi' = \Psi$. As Ψ is an S -morphism, it maps \mathcal{V}_K to $X \cong \mathcal{X}_K$. We have already seen that $\Phi|_{\mathcal{Z}_K} : \mathcal{Z}_K \rightarrow X$ is an isomorphism with inverse map $\Phi|_{\mathcal{Z}_K}^{-1}$. Therefore we have $\Psi'|_{\mathcal{V}_K} = \Phi|_{\mathcal{Z}_K}^{-1} \circ \Psi|_{\mathcal{V}_K}$. As \mathcal{V}_K is open and dense in \mathcal{V} , \mathcal{V} is reduced, and \mathcal{Z} is separated over \mathcal{X} , Ψ' is unique on \mathcal{V} by [GW10, Corollary 9.9]. Now we construct Ψ' . First we need to show some facts concerning \mathcal{Y} and the G -action on \mathcal{Y} . Consider $\mathcal{X}_T := \mathcal{X} \times_S T$ and the following diagram.

$$\begin{array}{ccccc}
 \mathcal{Y} & & & & \\
 \swarrow \varphi & & \searrow \pi & & \\
 & \mathcal{X}_T & \xrightarrow{p_{\mathcal{X}}} & \mathcal{X} & \\
 & \downarrow p_T & \square & \downarrow & \\
 & T & \xrightarrow{\pi_T} & S &
 \end{array}$$

Here $p_{\mathcal{X}}$ and p_T are the projection maps. Note that the diagram commutes, so there is a unique h with $p_T \circ h = \varphi$ and $p_{\mathcal{X}} \circ h = \pi$. As $p_{\mathcal{X}}$ and π are finite, by [GW10, Proposition 12.11] h is finite, too. As \mathcal{X} is flat over S , the G -action on T induces a G -action on \mathcal{X}_T such that p_T is G -equivariant and $\mathcal{X}_T/G = \mathcal{X}$, see [Gro63, Exposé V, Proposition 1.9]. As φ and p_T are G -equivariant, h is G -equivariant, too.

Let $n : \mathcal{X}_T^n \rightarrow \mathcal{X}_T$ be the normalization. By assumption, \mathcal{Y} is integral and smooth over T , so in particular normal. As $\mathcal{Y}_L = \mathcal{X}_L = \mathcal{X}_L$, h is generically an isomorphism, and therefore dominant. So the universal property of normalization induces a unique morphism $s : \mathcal{Y} \rightarrow \mathcal{X}_T^n$ such that $n \circ s = h$. Note that s is finite, because h and n are finite, and an isomorphism on $X_L \subset \mathcal{Y}$. Altogether s is a finite birational morphism between integral normal schemes. That means, by [GW10, Corollary 12.88] it is an isomorphism. So we may assume that $h = n$ and $\mathcal{Y} = \mathcal{X}_T^n$.

Back to \mathcal{V} and Ψ . Consider the following cartesian diagram.

$$\begin{array}{ccccc}
 & \mathcal{V}_T & \xrightarrow{\pi_{\mathcal{V}}} & \mathcal{V} & \\
 \Psi_T \swarrow & \downarrow p & \square & \downarrow \Psi & \\
 \mathcal{Y} & \xrightarrow{n} & \mathcal{X}_T & \longrightarrow & \mathcal{X}
 \end{array}$$

with $\mathcal{V}_T := \mathcal{V} \times_S T = \mathcal{V} \times_{\mathcal{X}} \mathcal{X}_T$, $\pi_{\mathcal{V}}$ and p the projection maps. As \mathcal{V} is smooth over S , so in particular flat, the G -action on T induces a G -action on \mathcal{V}_T such that

$\mathcal{V}_T \rightarrow T$ is G -equivariant and $\mathcal{V}_T/G = \mathcal{V}$, see [Gro63, Exposé V, Proposition 1.9]. By construction p is G -equivariant.

It might happen that \mathcal{V}_T is not connected. Let $\mathcal{V}_T = U_1 \sqcup \cdots \sqcup U_m$, with $U_i \subset \mathcal{V}_T$ the connected components. As $\mathcal{V} = \mathcal{V}_T/G$ is connected, G acts transitively on the connected components. As Ψ is dominant, the same holds for p . Note that \mathcal{X}_T is connected, because it is flat over T and generically isomorphic to the L -variety X_L . Hence there exists at least one component U_i such that $p|_{U_i}$ is dominant. As G acts transitively on \mathcal{V}_T and p is G -equivariant, $p|_{U_j}$ is dominant for every component U_j . By assumption \mathcal{V} is smooth over S , so \mathcal{V}_T is smooth over T . Hence every component U_i of \mathcal{V}_T is normal. So by the universal property of normalization there are unique morphisms $\Psi_T|_{U_i}: U_i \rightarrow \mathcal{Y}$ such that $n \circ \Psi_T|_{U_i} = p|_{U_i}$. This defines a unique morphism Ψ_T on all of \mathcal{V}_T such that $n \circ \Psi_T = p$. As p and n are G -equivariant, Ψ_T is G -equivariant, too.

Take any $W \in (\text{Sch}/S)$, $f \in \mathcal{V}(W)$. By the universal property of the fiber product, there is a unique $\tilde{f} \in \text{Hom}_T(W \times_S T, \mathcal{V}_T)$ such that $f \circ p_W = \pi_{\mathcal{V}} \circ \tilde{f}$.

One checks easily that \tilde{f} is G -equivariant. Set $\Psi'(f) := \Psi_T \circ \tilde{f}$. As Ψ_T is G -equivariant, $\Psi'(f) \in \mathcal{Z}(W)$. It is easy to check that this defines a map of functors, so we obtain an S -morphism $\Psi \in \text{Hom}_S(\mathcal{V}, \mathcal{Z})$.

We still need to check that $\Psi = \Phi \circ \Psi'$. Therefore it suffices to check that for all $f \in \mathcal{V}(W)$, $W \in (\text{Sch}/S)$ flat, $\Psi(f) = \Phi \circ \Psi'(f)$. Note that the following diagram commutes.

$$\begin{array}{ccccc}
 & & & & \mathcal{V} \\
 & & & \nearrow f & \searrow \pi_{\mathcal{V}} \\
 & & \mathcal{V}_T & \xrightarrow{\Psi_T} & \mathcal{Y} \\
 & \nearrow \tilde{f} & \searrow \Psi'(f) & \downarrow \pi & \downarrow \Psi \\
 W & \xleftarrow{p_W} & W \times_S T & \longrightarrow & T \\
 & \searrow & \downarrow & \downarrow \pi & \downarrow \pi \\
 & & S & \xleftarrow{\Phi(\Psi'(f))} & \mathcal{X}'
 \end{array}$$

One observes that $\Phi(\Psi'(f)) = \Psi \circ f = \Psi(f)$, which we wanted to show.

We still need to check that \mathcal{Z} is unique up to a unique isomorphism with its properties. Assume there is a \mathcal{Z}' and a morphism $\Phi' : \mathcal{Z}' \rightarrow \mathcal{X}$ having the same properties as \mathcal{Z} and Φ . So we get unique morphisms $\alpha : \mathcal{Z} \rightarrow \mathcal{Z}'$ and $\alpha' : \mathcal{Z}' \rightarrow \mathcal{Z}$ with $\Phi \circ \alpha' = \Phi'$ and $\Phi' \circ \alpha = \Phi$. Note that $\Phi \circ (\alpha' \circ \alpha) = \Phi' \circ \alpha = \Phi$. But $\text{id}_{\mathcal{Z}}$ is unique with $\Phi \circ \text{id}_{\mathcal{Z}} = \Phi$, so $\alpha' \circ \alpha = \text{id}_{\mathcal{Z}}$. Similarly one gets $\alpha \circ \alpha' = \text{id}_{\mathcal{Z}'}$. So α is the unique isomorphism over \mathcal{X} of \mathcal{Z} and \mathcal{Z}' .

The case that \mathcal{Y} is a weak Néron model. Assume that $\varphi : \mathcal{Y} \rightarrow T$ is a weak Néron model of X_L . Hence φ is separated, so by [Gro63, Exposé V, Proposition 1.5] \mathcal{X} is separated over S . As Φ is separated, $\mathcal{Z} \rightarrow S$ is separated, too. Hence to show that $\mathcal{Z} \rightarrow S$ is a weak Néron model of X , we still need to show that

$$\mathcal{Z}(S) = \text{Hom}_T(T, \mathcal{Y})^G \rightarrow \mathcal{Z}_K(K) = \text{Hom}_L(\text{Spec}(L), X_L)^G \cong X(K)$$

is a bijection. This map is injective, because \mathcal{Y} is a separated T -scheme. Take any $\sigma' \in \mathcal{Z}_K(K)$. As \mathcal{Y} is a weak Néron model of X_L , $\mathcal{Y}(T) \cong X_L(L)$, so there is a $\sigma \in \text{Hom}_T(T, \mathcal{Y})$ with $\sigma|_{\text{Spec}(L)} = \sigma'$. As g_T^{-1} maps $\text{Spec}(L)$ to itself, we get $g \circ \sigma \circ g_T^{-1}|_{\text{Spec}(L)} = \sigma|_{\text{Spec}(L)}$. As \mathcal{Y} is a separated T -scheme, $g \circ \sigma \circ g_T^{-1} = \sigma$, i.e. $\sigma \in \mathcal{Z}(S)$. Hence the map $\mathcal{Z}(S) \rightarrow \mathcal{Z}_K(K)$ is surjective. \square

In [Edi92, Theorem 4.2] the following statement is proven:

Theorem. *Let L/K be a tame Galois extension, \mathcal{O}_L the ring of integers of L , and $T := \text{Spec}(\mathcal{O}_L)$. Let X be an abelian variety over K . Then there is a good $\text{Gal}(L/K)$ -action on the Néron model $\varphi : \mathcal{Y} \rightarrow T$ of X_L extending the Galois action on X_L , and $\mathcal{Z} \rightarrow S$ given by Construction 3.1 is the Néron model of X .*

Note that the Néron model of an Abelian variety is uniquely determined by a universal property. In [Edi92] this universal property is used to show that \mathcal{Z} is the Néron model of X . As we do not have a universal property for weak Néron models in general, we had to use different methods to prove the universal property in Theorem 3.1.

Moreover, Néron models of Abelian Varieties are quasi-projective. In Theorem 3.1 we do not assume that \mathcal{Y} is quasi-projective, which makes the proof of the representability of \mathcal{Z} less straightforward.

Remark 3.2. If we do not assume that φ is smooth in Theorem 3.1, we can modify Construction 3.1 by considering $\text{Sm}(\mathcal{Y}/S)$, the smooth locus of \mathcal{Y} over S , instead of \mathcal{Y} . This is well defined, because the G -action restricts to $\text{Sm}(\mathcal{Y}/S)$. We will get a smooth model of X with an S -morphism Φ as in Theorem 3.1. Note that the map $\Phi(S) : \mathcal{Z}(S) \rightarrow \mathcal{X}(S)$ will be injective in this case, but in general not surjective. Nevertheless, if we assume that the smooth locus of \mathcal{Y} over S is a weak Néron model of X_L , the modified \mathcal{Z} will be a weak Néron model of X . This is in particular the case if \mathcal{Y} is regular and φ is proper.

Remark 3.3. If we do not assume in Theorem 3.1 that the Galois extension L/K is tame, we cannot show that \mathcal{Z} is smooth, because then [Edi92, Proposition 3.4] does not hold, see [Edi92, Example 4.3].

4. LOCAL STUDIES

4.1. Main Lemma.

Lemma 4.1. *Assumption and notation as in Theorem 3.1. Let \mathcal{Y}^G be the fixed locus of the G -action on \mathcal{Y} . Then there is a k -morphism*

$$b : \mathcal{Z}_k \rightarrow \mathcal{Y}^G$$

such that for every point $y \in \mathcal{Y}^G$ with residue field $\kappa(y)$ the inverse image of y is isomorphic to $\mathbb{A}_{\kappa(y)}^m$ as $\kappa(y)$ -schemes for some $m \in \mathbb{N}$.

Proof. As L/K is a tame Galois extension, $G = \mathbb{Z}/r\mathbb{Z}$ with r prime to $\text{char}(k)$. Let the G -action on \mathcal{Y} be given by $g \in \text{Aut}(\mathcal{Y})$, and that on T by $g_T \in \text{Aut}(T)$. By [Edi92, Proposition 3.1], \mathcal{Y}^G is a closed subscheme of \mathcal{Y} . Take any S -scheme W , $f \in \mathcal{Y}^G(W)$. Then $g_T \circ \varphi \circ f = \varphi \circ g \circ f = \varphi \circ f$, i.e. $\varphi \circ f \in T^G(W)$. As $T^G = \text{Spec}(k)$, \mathcal{Y}^G is a closed subscheme of $\mathcal{Y}_k \subset \mathcal{Y}$, so in particular a k -scheme of finite type.

To construct b , let $W \in (\text{Sch}/k)$ be any k -scheme, $w : W \rightarrow \text{Spec}(k)$ the structure map. Recall the construction of \mathcal{Z} in Construction 3.1. Set

$$b(W) : \mathcal{Z}_k(W) = \text{Hom}_T(W \times_S T, \mathcal{Y})^G \rightarrow \mathcal{Y}^G(W); f \mapsto f \circ i_W$$

with $i_W : W \hookrightarrow W \times_S T$ the inclusion of the special fiber. By construction, $b(W)(f) \in \text{Hom}_T(W, \mathcal{Y})$. We have $g \circ f \circ i_W = f \circ (\text{id}_W \times g_T) \circ i_W = f \circ i_W$. Here the first equation holds, because f is G -equivariant, and the second, because the action on the special fiber $i_W(W) \subset W \times_S T$ is trivial. Hence $b(W)(f) \in \mathcal{Y}^G(W)$. It is obvious that b is functorial, so we get the required k -morphism.

Let $y \in \mathcal{Y}^G$ be any point with residue field $\kappa(y)$, $j_y : \text{Spec}(\kappa(y)) \hookrightarrow \mathcal{Y}^G \subset \mathcal{Y}$ be the immersion of the point y . Note that $b^{-1}(y)$ is defined by the following cartesian diagram.

$$\begin{array}{ccc} b^{-1}(y) & \longrightarrow & \mathcal{Z}_k \\ \downarrow & \square & \downarrow b \\ \text{Spec}(\kappa(y)) & \xrightarrow{j_y} & \mathcal{Y}^G \end{array}$$

Take any affine $\kappa(y)$ -scheme $W = \text{Spec}(A) \in (\text{Sch}/\kappa(y))$ with structure map $\omega : W \rightarrow \text{Spec}(\kappa(y))$. By the universal property of the fiber product we obtain

$$\begin{aligned} b^{-1}(y)(W) &= \{f \in \mathcal{Z}_k(W) \mid b \circ f = j_y \circ \omega\} \\ &= \{f \in \text{Hom}_T(W \times_S T, \mathcal{Y})^G \mid f \circ i_W = j_y \circ \omega\}. \end{aligned}$$

As \mathcal{Y}^G is a subscheme of \mathcal{Y}_k , W is a k -scheme with structure map $\varphi \circ j_y \circ \omega$. Recall that G acts on $\text{Hom}_T(W \times_S T, \mathcal{Y})$ by sending $f \in \text{Hom}_T(W \times_S T, \mathcal{Y})$ to $g \circ f \circ (\text{id}_W \times g_T)^{-1}$. Set $R := \mathcal{O}_L$. Hence $R^G = \mathcal{O}_K$. We have

$$\begin{aligned} W \times_S T &= W \times_{\text{Spec}(k)} \text{Spec}(k) \times_S T = W \times_{\text{Spec}(k)} \text{Spec}(k \otimes_{R^G} R) \\ &\cong \text{Spec}(A \otimes_k k[t]/(t^r)) = \text{Spec}(A[t]/(t^r)). \end{aligned}$$

To compute $k \otimes_{R^G} R$ we use Lemma 7.5. This lemma also implies that

$$\alpha := (\text{id} \times g_T)^\# : A[t]/(t^r) \rightarrow A[t]/(t^r); p(t) \mapsto p(\mu t)$$

for a primitive r -th root of unity $\mu \in k \subset \kappa(y)$. Note that

$$r_W := i_W^\# : A[t]/(t^r) \rightarrow A; p(t) \mapsto p(0).$$

One observes that f sends all points in $\text{Spec}(A[t]/(t^r))$ to $y \in \mathcal{Y}$, so it factors uniquely through $\text{Spec}(\mathcal{O}_{\mathcal{Y},y})$, i.e. there is a unique morphism \tilde{f} such that the following diagram commutes.

$$\begin{array}{ccc} & \text{Spec}(\mathcal{O}_{\mathcal{Y},y}) & \\ \tilde{f} \nearrow & \downarrow j & \\ \text{Spec}(A[t]/(t^r)) & \xrightarrow{f} & \mathcal{Y} \end{array}$$

Let $r_y := i_y^\# : \mathcal{O}_{\mathcal{Y},y} \rightarrow \kappa(y)$ be the residue map. Note that $j \circ i_y = j_y$. As $f \circ i_W = j_y \circ \omega$, we get that $j \circ \tilde{f} \circ i_W = j \circ i_y \circ \omega$. The fact that j is a monomorphism implies that $\tilde{f} \circ i_W = i_y \circ \omega$.

As y lies in \mathcal{Y}^G , there is an induced G -action on $\text{Spec}(\mathcal{O}_{\mathcal{Y},y})$ given by some map $\tilde{g} \in \text{Aut}(\text{Spec}(\mathcal{O}_{\mathcal{Y},y}))$ with $\tilde{g}^r = \text{id}$ and $\alpha_y \in \text{Aut}(\mathcal{O}_{\mathcal{Y},y})$ with $\alpha_y^r = \text{id}$, respectively, such that j is G -equivariant. Hence $f = g \circ f \circ (\text{id}_W \times g_T)^{-1}$ implies that

$$j \circ (\tilde{g} \circ \tilde{f} \circ (\text{id}_W \times g_T)^{-1}) = g \circ j \circ \tilde{f} \circ (\text{id}_W \times g_T)^{-1} = f.$$

As \tilde{f} is unique with this property, $\tilde{g} \circ \tilde{f} \circ (\text{id}_W \times g_T)^{-1} = \tilde{f}$.

Assume that $\tilde{f} \in \text{Hom}_T(\text{Spec}(A[t]/(t^r)), \text{Spec}(\mathcal{O}_{\mathcal{Y},y}))$ with $\tilde{f} \circ i_W = i_y \circ \omega$, and $\tilde{g} \circ \tilde{f} \circ (\text{id}_W \times g_T)^{-1} = \tilde{f}$. Then $f := j \circ \tilde{f} \in \text{Hom}_T(\text{Spec}(A[t]/(t^r)), \mathcal{Y})$, and we have that $g \circ f \circ (\text{id}_W \times g_T)^{-1} = f$ as well as $f \circ i_W = j_y \circ \omega$. Altogether, we obtain

$$\begin{aligned} b^{-1}(y)(W) &= \{\tilde{f} \in \text{Hom}_T(\text{Spec}(A[t]/(t^r)), \text{Spec}(\mathcal{O}_{\mathcal{Y},y})) \\ &\quad \mid \tilde{f} \circ i_W = i_y \circ \omega \text{ and } \tilde{g} \circ \tilde{f} \circ (\text{id}_W \times g_T)^{-1} = \tilde{f}\} \\ &= \{a \in \text{Hom}_R(\mathcal{O}_{\mathcal{Y},y}, A[t]/(t^r)) \\ &\quad \mid r_W \circ a = \omega^\# \circ r_y \text{ and } \alpha^{-1} \circ a \circ \alpha_y = a\}. \end{aligned}$$

Now consider $\mathcal{O}_{\mathcal{Y},y}^G$. Note that, as $\mathcal{O}_{\mathcal{Y},y}$ is an R -module, it contains lifts of all roots of unity and hence by Remark 7.2 we get that Remark 7.4 and Lemma 7.5 hold. Let $i^G : \mathcal{O}_{\mathcal{Y},y}^G \hookrightarrow \mathcal{O}_{\mathcal{Y},y}$ be the inclusion, and $r_y^G : \mathcal{O}_{\mathcal{Y},y}^G \rightarrow \kappa(y)$ the residue map. We have $r_y^G = r_y \circ i^G$. Consider the following diagram.

$$(2) \quad \begin{array}{ccc} & & A[t]/(t^r) \\ & \nearrow a & \\ \mathcal{O}_{\mathcal{Y},y} & \xrightarrow{\rho_1} & \mathcal{O}_{\mathcal{Y},y} \otimes_{\mathcal{O}_{\mathcal{Y},y}^G} \kappa(y) \\ \uparrow i^G & & \uparrow \rho_2 \\ \mathcal{O}_{\mathcal{Y},y}^G & \xrightarrow{r_y^G} & \kappa(y) \end{array} \quad \begin{array}{c} \nearrow i_0 \circ \omega^\# \\ \searrow \tilde{a} \end{array}$$

Here $a \in b^{-1}(y)(W)$ as described before, ρ_1 and ρ_2 are the morphisms we get from the definition of tensor product, and $i_0 : A \rightarrow A[t]/(t^r); c \mapsto c$. Note that $r_W \circ i_0 = \text{id}$, and $i_0 \circ r_W|_{i_0(A)} = \text{id}$. One observes that for every $u \in \mathcal{O}_{\mathcal{Y},y}^G$, $(\alpha^{-1} \circ a)(u) = (\alpha^{-1} \circ a \circ \alpha_y)(u) = a(u)$. Set $a(u) = \sum_{i=0}^{r-1} a_i t^i$ for some $a_i \in A$. Hence $(\alpha^{-1} \circ a)(u) = \sum_{i=0}^{r-1} \mu^{-i} a_i t^i$. Comparing coefficients yields $a(u) = a_0$, i.e. $a(\mathcal{O}_{\mathcal{Y},y}^G) \subset i_0(A)$. Using in addition that $r_y^G = r_y \circ i^G$ and $r_W \circ a = \omega^\# \circ r_y$, we obtain

$$i_0 \circ \omega^\# \circ r_y^G = i_0 \circ \omega^\# \circ r_y \circ i^G = i_0 \circ r_W \circ a \circ i^G = i_0 \circ r_W|_{i_0(A)} \circ a \circ i^G = a \circ i^G.$$

Hence by the universal property of tensor product there is a unique \tilde{a} such that diagram (2) commutes.

Now, G acts on $\mathcal{O}_{\mathcal{Y},y} \otimes_{\mathcal{O}_{\mathcal{Y},y}^G} \kappa(y)$ given by $\tilde{\alpha}_y \in \text{Aut}(\mathcal{O}_{\mathcal{Y},y} \otimes_{\mathcal{O}_{\mathcal{Y},y}^G} \kappa(y))$, such that ρ_1 and ρ_2 are G -equivariant, see Lemma 7.5. As $\alpha^{-1} \circ a \circ \alpha_y = a$, we get

$$(\alpha^{-1} \circ \tilde{a} \circ \tilde{\alpha}_y) \circ \rho_1 = \alpha^{-1} \circ \tilde{a} \circ \rho_1 \circ \alpha_y = a$$

and, using that G acts trivially on $i_0(A)$, we obtain

$$(\alpha^{-1} \circ \tilde{a} \circ \tilde{\alpha}_y) \circ \rho_2 = \alpha^{-1} \circ \tilde{a} \circ \rho_2 = \alpha^{-1} \circ i_0 \circ \omega^\# = i_0 \circ \omega^\#.$$

As \tilde{a} is unique with these properties, $\alpha^{-1} \circ \tilde{a} \circ \tilde{\alpha}_y = \tilde{a}$.

Denote by $\tilde{r} : k \otimes_{R^G} R \cong k[t]/(t^r) \rightarrow A \otimes_k k \otimes_{R^G} R \cong A[t]/(t^r)$ the canonical map given by the properties of the tensor product. We have $\tilde{r}(t) = t$. The R -structure of $A[t]/(t^r)$ is given by $\tilde{r} \circ \rho_R$, with $\rho_R : R \rightarrow k \otimes_{R^G} R$ the canonical map. The R -structure of $\mathcal{O}_{\mathcal{Y},y}$ is given by $\beta_y := (\varphi \circ j)^\#$. As a is an R -morphism, we obtain the following commutative diagram.

$$(3) \quad \begin{array}{ccccc} & & k[t]/(t^r) & \xleftarrow{\rho_R} & R \\ & \nearrow \tilde{r} & \downarrow \tilde{\beta}_y & & \downarrow \beta_y \\ A[t]/(t^r) & \xleftarrow{\tilde{a}} & \mathcal{O}_{\mathcal{Y},y} \otimes_{\mathcal{O}_{\mathcal{Y},y}^G} \kappa(y) & \xleftarrow{\rho_1} & \mathcal{O}_{\mathcal{Y},y} \\ & \searrow a & & & \end{array}$$

By Remark 7.4, $R^G \subset R$ is a local subring having the same residue field as R . As β_y is G -equivariant, it maps R^G to $\mathcal{O}_{\mathcal{Y},y}^G$.

It is easy to check that the following diagram commutes.

$$\begin{array}{ccccc}
\kappa(y) \otimes_{\mathcal{O}_{\mathcal{Y},y}^G} \mathcal{O}_{\mathcal{Y},y} & \xleftarrow{\rho_1} & \mathcal{O}_{\mathcal{Y},y} & & \\
\uparrow \rho_2 & \nwarrow \tilde{\beta}_y & \nwarrow \beta_y & \nwarrow i^G & \\
\kappa(y) & & k \otimes_R R & \xleftarrow{\rho_R} & R \\
& \nearrow \tilde{r}_y & \uparrow & \nearrow \beta_y|_{R^G} & \\
& & k & \xleftarrow{r_y^G} & R^G \\
& & & & \searrow \beta_y|_{R^G} \\
& & & & \mathcal{O}_{\mathcal{Y},y}^G
\end{array}$$

Hence the universal property of tensor product induces a unique k -morphism $\tilde{\beta}_y$ with $\tilde{\beta}_y \circ \rho_R = \rho_1 \circ \beta_y$. Looking at diagram (3) again, we get

$$\tilde{r} \circ \rho_R = a \circ \beta_y = \tilde{a} \circ \rho_1 \circ \beta_y = \tilde{a} \circ \tilde{\beta}_y \circ \rho_R.$$

As ρ_R is surjective, $\tilde{r} = \tilde{a} \circ \tilde{\beta}_y$, i. e. \tilde{a} preserves the $k[t]/(t^r)$ -structure given by $\tilde{\beta}_y$ on $\mathcal{O}_{\mathcal{Y},y} \otimes_{\mathcal{O}_{\mathcal{Y},y}^G} \kappa(y)$, and on $A[t]/(t^r)$ given by \tilde{r} .

Using the universal property of the tensor product, and that $r_y \circ i^G = r_y^G$, we get a unique morphism $\tilde{r}_y : \mathcal{O}_{\mathcal{Y},y} \otimes_{\mathcal{O}_{\mathcal{Y},y}^G} \kappa(y) \rightarrow \kappa(y)$, such that $\tilde{r}_y \circ \rho_1 = r_y$ and $\tilde{r}_y \circ \rho_2 = \text{id}$.

Using that $r_W \circ a = \omega^\# \circ r_y$, we get

$$\begin{aligned}
(r_W \circ \tilde{a}) \circ \rho_1 &= r_W \circ a \text{ and } (\omega^\# \circ \tilde{r}_y) \circ \rho_1 = \omega^\# \circ r_y = r_W \circ a, \\
(r_W \circ \tilde{a}) \circ \rho_2 &= r_W \circ i_0 \circ \omega^\# = \omega^\# \text{ and } (\omega^\# \circ \tilde{r}_y) \circ \rho_2 = \omega^\#.
\end{aligned}$$

Moreover, $r_W \circ a \circ i^G = r_W \circ i_0 \circ \omega^\# \circ r_y^G = \omega^\# \circ r_y^G$, hence by the universal property of the tensor product there is a unique morphism $v : \mathcal{O}_{\mathcal{Y},y} \otimes_{\mathcal{O}_{\mathcal{Y},y}^G} \kappa(y) \rightarrow A$ such that $v \circ \rho_1 = r_W \circ a$ and $v \circ \rho_2 = \omega^\#$, in particular $r_W \circ \tilde{a} = v = \omega^\# \circ \tilde{r}_y$, meaning that \tilde{a} is a $\kappa(y)$ -morphism.

Note that for a given morphism $\tilde{a} \in \text{Hom}_{k[t]/(t^r)}(\mathcal{O}_{\mathcal{Y},y} \otimes_{\mathcal{O}_{\mathcal{Y},y}^G} \kappa(y), A[t]/(t^r))$, we have that $a := \tilde{a} \circ \rho_1 \in \text{Hom}_R(\mathcal{O}_{\mathcal{Y},y}, A[t]/(t^r))$. If $\alpha^{-1} \circ \tilde{a} \circ \tilde{\alpha}_y = \tilde{a}$, then $\alpha^{-1} \circ a \circ \alpha_y = a$. If we assume furthermore that $r_W \circ \tilde{a} = \omega^\# \circ \tilde{r}_y$, then $r_W \circ a = \omega^\# \circ r_y$. So altogether

$$\begin{aligned}
b^{-1}(y)(W) &= \{\tilde{a} \in \text{Hom}_{k[t]/(t^r)}(\mathcal{O}_{\mathcal{Y},y} \otimes_{\mathcal{O}_{\mathcal{Y},y}^G} \kappa(y), A[t]/(t^r)) \\
&\quad | \tilde{a} \circ \rho_2 = i_0 \circ \omega^\# \text{ and } r_W \circ \tilde{a} = \omega^\# \circ \tilde{r}_y \text{ and } \alpha^{-1} \circ \tilde{a} \circ \tilde{\alpha}_y = \tilde{a}\}.
\end{aligned}$$

Note that $\tilde{a} \circ \rho_2 = i_0 \circ \omega^\#$ is actually redundant.

By Lemma 7.5, $\mathcal{O}_{\mathcal{Y},y} \otimes_{\mathcal{O}_{\mathcal{Y},y}^G} \kappa(y) \cong \kappa(y)[x_0, \dots, x_m]/\mathfrak{I}$,

$$\tilde{\alpha}_y(p(x_0, \dots, x_m)) = p(\mu^{\ell_0} x_0, \dots, \mu^{\ell_m} x_m)$$

for $p(x_0, \dots, x_m) \in \kappa(y)[x_0, \dots, x_m]/\mathfrak{I}$, $\mu \in k \subset \kappa(y)$ a primitive r -th root of unity, $\ell_i \in \{1, \dots, r-1\}$, $m \in \mathbb{N}$, and $\mathfrak{I} \subset \kappa(y)[x_0, x_1, \dots, x_m]$ is the ideal generated by monomials of the form $x_0^{s_0} \dots x_m^{s_m}$ such that $\ell_0 s_0 + \dots + \ell_m s_m = rs$, $s \in \mathbb{N}$.

We now want to show that we can assume that $x_0 = \tilde{\beta}_y(t)$. By Lemma 7.1 there is a local parameter $t' \in R$ with $g_T^\#(t') = \mu t'$, $\mu \in R$ a primitive r -th root of unity. Looking at the proof of Lemma 7.5 we may assume that $\rho_R(t') = t$, i.e. $\tilde{\beta}_y(t) = \tilde{\beta}_y \circ \rho_R(t') = \rho_1 \circ \beta_y(t')$. So looking again at the proof of Lemma 7.5 it suffices to find a regular system of parameters $y_0, \dots, y_{m'} \in \mathcal{O}_{\mathcal{Y},y}$ such that $\alpha_y(y_i) = \mu^{\ell_i} y_i$, $\ell_i \in \{0, \dots, r-1\}$, $\ell_0 \neq 0$, and $y_0 = \beta_y(t') =: \tilde{t}$. Note that $\alpha_y(\tilde{t}) = \beta_y(g_T^\#(t')) = \mu \tilde{t}$. Hence using Lemma 7.1 it suffices to show that $\tilde{t} \in \mathfrak{m}_y$,

and $\tilde{t} \neq 0 \pmod{\mathfrak{m}_y^2}$ for the maximal ideal $\mathfrak{m}_y \subset \mathcal{O}_{\mathcal{Y},y}$. As y lies in $\mathcal{Y}^G \subset \mathcal{Y}_k$, $\tilde{t} \in \mathfrak{m}_y$. Let $U = \text{Spec}(C) \subset \mathcal{Y}$ be an affine neighborhood of y , $\mathfrak{p} \subset C$ the defining prime ideal of y . Choose a maximal ideal $\mathfrak{m} \subset C$ with $\mathfrak{p} \subset \mathfrak{m}$, let y' be the corresponding closed point. By [Gro67, Proposition 17.5.3], $\hat{\mathcal{O}}_{\mathcal{Y},y'} \cong R[[\tilde{y}_1, \dots, \tilde{y}_n]]$ as R -module. So $\tilde{t} \neq 0 \in \hat{\mathcal{O}}_{\mathcal{Y},y'}/\mathfrak{m}^2 \cong \mathcal{O}_{\mathcal{Y},y'}/\mathfrak{m}^2$. As $\mathfrak{p} \subset \mathfrak{m}$, $\tilde{t} \neq 0 \in \mathcal{O}_{\mathcal{Y},y'}/\mathcal{O}_{\mathcal{Y},y'}\mathfrak{p}^2$. As $\mathcal{O}_{\mathcal{Y},y'}/\mathcal{O}_{\mathcal{Y},y'}\mathfrak{p}^2 \subset \mathcal{O}_{\mathcal{Y},y}/\mathfrak{m}_y^2$ as R -modules, we have $\tilde{t} \neq 0 \pmod{\mathfrak{m}_y^2}$. This is what we wanted to show, so we may assume that $\tilde{\beta}_y$ is the k -morphism sending $t \in k[t]/(t^r)$ to x_0 .

Now chose any $\tilde{a} \in b^{-1}(y)(W)$. For $j \in \{1, \dots, m\}$ we have

$$\tilde{a}(x_j) = \sum_{i=0}^{r-1} a_{ij} t^i \in A[t]/(t^r)$$

for some $a_{ij} \in A$. Using $r_W \circ \tilde{a} = \omega^\# \circ \tilde{r}_y$, we obtain

$$a_{0j} = r_W(\tilde{a}(x_j)) = \omega^\#(\tilde{r}_y(x_j)) = \omega^\#(0) = 0.$$

From $\alpha^{-1} \circ \tilde{a} \circ \tilde{\alpha} = \tilde{a}$ we get

$$\sum_{i=1}^{r-1} \mu^{\ell_j-i} a_{ij} t^i = (\alpha^{-1} \circ \tilde{a} \circ \tilde{\alpha}_y)(x_j) = \tilde{a}(x_j) = \sum_{i=1}^{r-1} a_{ij} t^i.$$

Comparing coefficients yields that either $a_{ij} = 0$, or $\mu^{\ell_j-i} = 1$. As i and ℓ_i lie in $\{1, \dots, r-1\}$, the latter is equivalent to $i = \ell_j$. As \tilde{a} preserves the $k[t]/(t^r)$ -structure, i.e. $\tilde{a} \circ \tilde{\beta}_y = \tilde{r}$, we get that $\tilde{a}(x_0) = t$. So using that \tilde{a} is a $\kappa(y)$ -morphism, i.e. that $\tilde{a} \circ \rho_2 = i_0 \circ w^\#$, we get that

$$(4) \quad \tilde{a}(p(x_0, x_1, \dots, x_m)) = p(t, a_1 t^{\ell_1}, \dots, a_m t^{\ell_m})$$

for all $p(x_0, x_1, \dots, x_m) \in \kappa(y)[x_0, x_1, \dots, x_m]/\mathfrak{I}$, and for some $a_i \in A$.

Let $\tilde{a} : \kappa(y)[x_0, x_1, \dots, x_m] \rightarrow A[t]/(t^r)$ be defined by formula (4). For any generator $x_0^{s_0} x_1^{s_1} \dots x_m^{s_m}$ of \mathfrak{I}

$$\tilde{a}(x_0^{s_0} x_1^{s_1} \dots x_m^{s_m}) = a_1^{s_1} \dots a_m^{s_m} t^{s_0 + \ell_1 s_1 + \dots + \ell_m s_m} = t^{rs} = 0 \in A[t]/(t^r).$$

This implies that $\mathfrak{I} \subset \ker(\tilde{a})$. Therefore, we get a unique well-defined map

$$\tilde{a} : \kappa(y)[x_0, x_1, \dots, x_m]/\mathfrak{I} \rightarrow A[t]/(t^r).$$

Note that \tilde{a} is a $\kappa(y)$ -morphism, and preserves the $k[t]/(t^r)$ -structure, and one can check that $\alpha^{-1} \circ \tilde{a} \circ \tilde{\alpha}_y = \tilde{a}$, and $r_W \circ \tilde{a} = w^\# \circ \tilde{r}_y$. Altogether, $\tilde{a} \in b^{-1}(y)(W)$ if and only if it is given by formula (4).

Now we are ready to construct a $\kappa(y)$ -isomorphism $\beta : b^{-1}(y) \rightarrow \mathbb{A}_{\kappa(y)}^m$. It suffices to give bijective, functorial maps

$$\beta(W) : b^{-1}(y)(W) \rightarrow \mathbb{A}_{\kappa(y)}^m(W) = \text{Hom}_{\kappa(y)}(\kappa(y)[y_1, \dots, y_m], A)$$

for all affine $W = \text{Spec}(A) \in (\text{Sch}/\kappa(y))$. Let $\beta(W)$ be the map which sends $\tilde{a} \in b^{-1}(W)$ given by formula (4) to $a' \in \mathbb{A}_{\kappa(y)}^m(W)$ with

$$a' : \kappa(y)[y_1, \dots, y_m] \rightarrow A; p(y_1, \dots, y_m) \mapsto p(a_1, \dots, a_m).$$

This map is bijective, because there is an obvious inverse map. It is easy to check that it is functorial. \square

Remark 4.2. If we do not assume that the Galois extension L/K is tame, then we cannot show Lemma 4.1. This is because Lemma 7.1 is wrong in this case, see Example 7.3, and hence we cannot show Lemma 7.5, which is the main ingredient of the proof of Lemma 4.1. It would be very interesting to know what happens in the non-tame case.

Remark 4.3. Note that if we do not assume that $k = \bar{k}$, but that L over K is totally ramified, and that k contains all r -th primitive roots of unity, one can still show Lemma 4.1.

There should also be no problem to replace \mathcal{O}_K by a Henselian ring.

4.2. Sections of the Quotient.

Corollary 4.4. *Assumptions and notation as in Theorem 3.1. Then $\mathcal{X}(\mathcal{O}_K) = \emptyset$ if and only if $\mathcal{Y}^G = \emptyset$. If $\mathcal{Y} \rightarrow T$ is a weak Néron model of X_L , then $X(K) = \emptyset$ if and only if $\mathcal{Y}^G = \emptyset$.*

Proof. By Theorem 3.1, $\mathcal{X}(\mathcal{O}_K) \cong \mathcal{Z}(\mathcal{O}_K)$. As $\mathcal{Z} \rightarrow S$ is smooth and \mathcal{O}_K is Henselian, $\mathcal{Z}(\mathcal{O}_K) = \emptyset$ if and only if $\mathcal{Z}_k(k) = \emptyset$ by [BLR90, Chapter 2.3, Proposition 5]. As k is algebraically closed this is equivalent to $\mathcal{Z}_k = \emptyset$. By Lemma 4.1 there is a surjective morphism $b : \mathcal{Z}_k \rightarrow \mathcal{Y}^G$, so $\mathcal{Z}_k = \emptyset$ if and only if $\mathcal{Y}^G = \emptyset$.

If $\mathcal{Y} \rightarrow T$ is a weak Néron model of X_L , $\mathcal{Z} \rightarrow S$ is a weak Néron model of X by Theorem 3.1, i.e. in particular $\mathcal{Z}(\mathcal{O}_K) \cong X(K)$, which implies the second claim. \square

Now we show one direction of the equivalence in Corollary 4.4 without using \mathcal{Z} , because this alternative proof yields an explicit construction of a section of the model $\mathcal{X} \rightarrow S$ through the image of a fixed point.

Proposition 4.5. *Assumptions and notation as in Theorem 3.1. If $\mathcal{Y}^G \neq \emptyset$, then $\mathcal{X}(\mathcal{O}_K) \neq \emptyset$.*

Proof. As $\mathcal{Y}^G \neq \emptyset$, there is a closed fixed point $y \in \mathcal{Y}$. Note that G acts on $\text{Spec}(\hat{\mathcal{O}}_{\mathcal{Y},y})$ given by some $\alpha_y \in \text{Aut}(\hat{\mathcal{O}}_{\mathcal{Y},y})$ with $\alpha_y^r = \text{id}$, such that the natural map $j : \text{Spec}(\hat{\mathcal{O}}_{\mathcal{Y},y}) \rightarrow \mathcal{Y}$ is G -equivariant. As L/K is a tame Galois extension, G is a cyclic group of order prime to $\text{char}(k)$, so by Lemma 7.1, G acts on $R := \mathcal{O}_L$ given by some $\alpha_R \in \text{Aut}(R)$ sending a generator t of the maximal ideal in R to μt , with $\mu \in R$ a primitive r -th root of unity. Note that $\hat{\mathcal{O}}_{\mathcal{Y},y}$ is an R -module via $\beta_y := (\varphi \circ j)^\#$, and β_y is G -equivariant. As φ is smooth, and the residue field of R is equal to the residue field of $\hat{\mathcal{O}}_{\mathcal{Y},y}$, [Gro67, Proposition 17.5.3] implies that $\hat{\mathcal{O}}_{\mathcal{Y},y} \cong R[\tilde{x}_1, \dots, \tilde{x}_n]$ as R -module for some $\tilde{x}_1, \dots, \tilde{x}_n \in \hat{\mathcal{O}}_{\mathcal{Y},y}$. Note that $t, \tilde{x}_1, \dots, \tilde{x}_n$ form a regular system of parameters of $\hat{\mathcal{O}}_{\mathcal{Y},y}$. As $\alpha_y(t) = \alpha_R(t) = \mu t$, by Lemma 7.1 we may choose a system of parameters x_0, \dots, x_n with $\alpha_y(x_i) = \mu^{\ell_i} x_i$ for some $\ell_i \in \mathbb{N}$, such that $x_0 = t$. So $\hat{\mathcal{O}}_{\mathcal{Y},y} \cong R[\tilde{x}_1, \dots, \tilde{x}_n] \cong R[x_1, \dots, x_n]$ as R -modules. Let $I \subset \hat{\mathcal{O}}_{\mathcal{Y},y}$ be the ideal generated by x_1, \dots, x_n . Note that $\alpha_y(I) \subset I$. So the quotient map

$$\hat{\sigma} : \hat{\mathcal{O}}_{\mathcal{Y},y} \rightarrow \hat{\mathcal{O}}_{\mathcal{Y},y}/I = R[x_1, \dots, x_n]/(x_1, \dots, x_n) \cong R$$

is a G -equivariant retraction of β_y . Therefore $\hat{\sigma}^\#$ is a section of $\varphi \circ j$, and $\sigma := j \circ \hat{\sigma}^\#$ is a section of φ . As both $\hat{\sigma}$ and j are G -equivariant, the same holds for σ .

Let the G -action on T be given by $g_T \in \text{Aut}(T)$, that on \mathcal{Y} by $g \in \text{Aut}(\mathcal{Y})$. Let $\pi : \mathcal{Y} \rightarrow \mathcal{X}$ and $\pi_T : T \rightarrow S$ be the quotients. Let $\varphi_G : \mathcal{X} \rightarrow S$ be the structure map of \mathcal{X} as S -scheme. Every element in $\mathcal{X}(\mathcal{O}_K)$ is given by a section of φ_G . As σ is G -invariant and π is a quotient map, $\pi \circ \sigma \circ g_T = \pi \circ g \circ \sigma = \pi \circ \sigma$. So by the universal property of the quotient $\pi_T : T \rightarrow S$, there exists a unique $\sigma_G : S \rightarrow \mathcal{X}$ such that $\pi \circ \sigma = \sigma_G \circ \pi_T$. Furthermore,

$$\varphi_G \circ \sigma_G \circ \pi_T = \varphi_G \circ \pi \circ \sigma = \pi_T \circ \varphi \circ \sigma = \pi_T \circ \text{id}_T = \pi_T.$$

As π_T is an epimorphism, $\varphi_G \circ \sigma_G = \text{id}_S$, i.e. σ_G is a section of φ_G . \square

Note that the image of a closed fixed point $y \in \text{Sm}(\mathcal{Y}/T)^G$ in \mathcal{X} is a singular point in general, so in fact we construct sections through singular points. Here is an example for such a section through a singular point.

Example 4.6. Assumptions and notation as in Example 2.7. The closed point $Q = (0, 0) \in \mathcal{Y} = \mathbb{A}_{k[[t]]}^1$ is fixed, and the $k[[s]]$ -scheme

$$\mathcal{Y}/G \cong \operatorname{Spec}(k[[s]][b, c]/(sc - b^2))$$

is singular in the image $Q' = (0, 0, 0)$ of Q under the quotient map. The proof of Proposition 4.5 implies that there is a section σ_G of $\mathcal{Y}/G \rightarrow \operatorname{Spec}(k[[s]])$ through Q' . Such a section is for example given by

$$\sigma_G^\#(P(s, b, c)) = P(s, 0, 0) \in k[[s]].$$

Note that the G -equivariant section σ of $\mathcal{Y} \rightarrow \operatorname{Spec}(k[[t]])$ which descends to σ_G is given by $\sigma^\#(P(t, x)) = P(t, 0)$.

5. MOTIVIC INVARIANTS

5.1. Motivic Serre Invariant.

Definition 5.1. The *Grothendieck group of k -varieties* $K_0(\operatorname{Var}_k)$ is defined to be the abelian group with

- generators: isomorphism classes $[U]$ of separated k -schemes U of finite type
- relations: $[U] = [U \setminus V] + [V]$ for every closed immersion $V \hookrightarrow U$ (*scissor relations*)

The product $[U][V] = [U \times_{\operatorname{Spec}(k)} V]$ defines a ring structure on $K_0(\operatorname{Var}_k)$. We call this ring the *Grothendieck ring of k -varieties*.

Set $\mathbb{L} := [\mathbb{A}_k^1]$.

The *modified Grothendieck ring of k -varieties* $K_0^{\operatorname{mod}}(\operatorname{Var}_k)$ is the quotient of $K_0(\operatorname{Var}_k)$ by the ideal generated by elements

$$[U] - [V]$$

where U and V are separated k -schemes of finite type such that there exists a finite, surjective, purely inseparable k -morphism $U \rightarrow V$.

We still denote the image of \mathbb{A}_k^1 in $K_0^{\operatorname{mod}}(\operatorname{Var}_k)$ by \mathbb{L} .

$$K_0^{\mathcal{O}_K}(\operatorname{Var}_k) := \begin{cases} K_0(\operatorname{Var}_k) & \text{if } \mathcal{O}_K \text{ has equal characteristic} \\ K_0^{\operatorname{mod}}(\operatorname{Var}_k) & \text{if } \mathcal{O}_K \text{ has mixed characteristic} \end{cases}$$

Definition 5.2. Let X be a smooth K -variety with weak Néron model $\mathcal{X} \rightarrow S$. Then the *motivic Serre invariant* $S(X)$ is defined by

$$S(X) := [\mathcal{X}_k] \in K_0^{\mathcal{O}_K}(\operatorname{Var}_k)/(\mathbb{L} - 1).$$

By [NS11a, Proposition-Definition 3.6] this definition does not depend on the choice of a weak Néron model.

Remark 5.1. Let X be a smooth, separated K -variety without K -rational point. Then $S(X) = 0$. This holds, because in this case X viewed as an S -scheme is a weak Néron model of X , i. e. the special fiber of this weak Néron model is empty. Hence if $S(X) \neq 0$, then X has a K -rational point.

Theorem 5.2. Let X be a smooth, proper K -variety. Let L/K be a tame Galois extension, \mathcal{O}_L the ring of integers of L , $T := \operatorname{Spec}(\mathcal{O}_L)$. Let $\varphi : \mathcal{Y} \rightarrow T$ be a weak Néron model of X_L with a good $G := \operatorname{Gal}(L/K)$ -action extending the Galois action on X_L . Then

$$S(X) = [\mathcal{Y}^G] \in K_0^{\mathcal{O}_K}(\operatorname{Var}_k)/(\mathbb{L} - 1).$$

Proof. By Theorem 3.1 we know that $\mathcal{Z} \rightarrow S$ as constructed in Construction 3.1 is a weak Néron model of X . Hence by definition $S(X)$ equals to the class of the special fiber \mathcal{Z}_k in $K_0^{\mathcal{O}_K}(\text{Var}_k)/(\mathbb{L} - 1)$, and it suffices to show the following statement:

$$[\mathcal{Z}_k] = [\mathcal{Y}^G] \in K_0^{\mathcal{O}_K}(\text{Var}_k)/(\mathbb{L} - 1).$$

As $K_0^{\mathcal{O}_K}(\text{Var}_k)/(\mathbb{L} - 1)$ is a quotient of $K_0(\text{Var}_k)/(\mathbb{L} - 1)$, it suffices to show the equation in $K_0(\text{Var}_k)/(\mathbb{L} - 1)$.

Consider $b : \mathcal{Z}_k \rightarrow \mathcal{Y}^G$ as in Lemma 4.1. We can find $U_i \subset \mathcal{Y}^G$ such that $\mathcal{Y}^G = U_1 \sqcup \dots \sqcup U_m$ with $U_i \subset \mathcal{Y}^G \setminus (\cap_{j < i} U_j)$ open, and $b^{-1}(U_i) \cong \mathbb{A}_{U_i}^{m_i}$ for some $m_i \in \mathbb{N}$, by proceeding in the following way.

By [Edi92, Proposition 3.5], \mathcal{Y}^G is smooth over $T^G = \text{Spec}(k)$, hence in particular reduced. Replacing \mathcal{Y}^G by an open subset, we may assume that it is integral. Let η be the generic point of \mathcal{Y}^G with residue field $\kappa(\eta)$. By Lemma 4.1 there is an isomorphism $\beta : b^{-1}(\eta) \rightarrow \mathbb{A}_{\kappa(\eta)}^{m_1}$ over $\kappa(\eta)$ for some $m_1 \in \mathbb{N}$. As β is defined by finitely many rational functions over \mathcal{Y}^G , we find an open subset U_1 of \mathcal{Y}^G over which β is already defined. In particular $b^{-1}(U_1) \cong \mathbb{A}_{U_1}^{m_1}$. Now one can proceed with $\mathcal{Y}^G \setminus U_1$ in the same way. The claim follows by noetherian induction using that \mathcal{Y}^G is of finite type over k .

So using the scissor relations in the Grothendieck ring of k -varieties we get in $K_0(\text{Var}_k)/(\mathbb{L} - 1)$ that

$$\begin{aligned} [\mathcal{Z}_k] &= [b^{-1}(\mathcal{Y}^G)] = [b^{-1}(U_1) \sqcup \dots \sqcup b^{-1}(U_m)] \\ &= [b^{-1}(U_1)] + [b^{-1}(U_2) \sqcup \dots \sqcup b^{-1}(U_m)] = \dots \\ &= \sum_{i=1}^m [b^{-1}(U_i)] = \sum_{i=1}^m [\mathbb{A}_{U_i}^{m_i}] = \sum_{i=1}^m [\mathbb{A}_k^{m_i}][U_i] = \sum_{i=1}^m [U_i] \\ &= [U_1 \sqcup \dots \sqcup U_m] = [\mathcal{Y}^G]. \end{aligned}$$

This proves Theorem 5.2. □

5.2. Rational Volume.

Fact. [NS11b, Example 4.3 and Corollary 4.14]

There exists a unique ring morphism (realization morphism)

$$\chi_c : K_0^{\mathcal{O}_K}(\text{Var}_k)/(\mathbb{L} - 1) \rightarrow \mathbb{Z}$$

that sends a class of a separated k -scheme U of finite type to the Euler characteristic with proper support

$$\chi_c(U) = \sum_{i \geq 0} (-1)^i \dim H_c^i(U, \mathbb{Q}_l),$$

with $l \neq \text{char}(k)$ a prime. The map does not depend on the choice of l .

Definition 5.3. Let X be a smooth K -variety with weak Néron model. Then the *rational volume* of X is defined by

$$s(X) := \chi_c(S(X)) \in \mathbb{Z}.$$

Remark 5.3. Let X be a smooth K -variety without K -rational point. Then $s(X) = 0$. This holds, because by Remark 5.1, $S(X) = 0$, hence in particular $s(X) = \chi_c(S(X)) = 0$. So if $s(X) \neq 0$, then X has a K -rational point.

Theorem 5.4. Let X be a smooth, proper K -variety, and let L/K be a tame Galois extension, such that $G := \text{Gal}(L/K)$ is a q -group, $q \neq \text{char}(k)$ a prime. Then

$$s(X_L) = s(X) \pmod{q}.$$

In particular, if $s(X_L)$ does not vanish modulo q , then X has a K -rational point.

Proof. Let \mathcal{O}_L be the ring of integers of L , $T := \text{Spec}(\mathcal{O}_L)$. By Theorem 2.9 there is a weak Néron model $\varphi : \mathcal{Y} \rightarrow T$ of X_L with a good G -action on \mathcal{Y} , extending the Galois action on X_L . Hence Theorem 5.2 implies that

$$S(X) = [\mathcal{Y}^G] \in K_0^{\mathcal{O}_K}(\text{Var}_k)/(\mathbb{L} - 1).$$

As $X_L \subset \mathcal{Y}$ is G -invariant, the same holds for \mathcal{Y}_k , so the action of G on \mathcal{Y} restricts to \mathcal{Y}_k . By [EN11, Proposition 5.4], for every variety U over a field F with a good G -action, $\chi_c(U) = \chi_c(U^G) \pmod{q}$. This Proposition is based on an argument in [Ser09, Section 7.2]. In our case we get

$$\chi_c(\mathcal{Y}_k) = \chi_c(\mathcal{Y}_k^G) \pmod{q}.$$

As $\mathcal{Y}^G \subset \mathcal{Y}_k$, see the proof of Lemma 4.1, $\mathcal{Y}^G = \mathcal{Y}_k^G$. As \mathcal{Y} is a weak Néron model of X_L , by definition $S(X_L) = [\mathcal{Y}_k] \in K_0^{\mathcal{O}_K}(\text{Var}_k)$. So altogether we obtain

$$\begin{aligned} s(X_L) &= \chi_c(S(X_L)) = \chi_c(\mathcal{Y}_k) = \chi_c(\mathcal{Y}_k^G) \pmod{q} \\ &= \chi_c(S(X)) \pmod{q} \\ &= s(X) \pmod{q}. \end{aligned}$$

Assume now that $s(X_L) \neq 0 \pmod{q}$. This implies that $s(X) \neq 0$. But the rational volume of a smooth K -variety without K -rational point vanishes, see Remark 5.3, hence X has a K -rational point. \square

6. RATIONAL POINTS ON CERTAIN VARIETIES WITH POTENTIAL GOOD REDUCTION

Definition 6.1. A smooth, proper K -variety X has *potential good reduction* (after a base change of order r) if there exists a Galois extension L/K (of degree r), such that X_L has a smooth and proper model.

Corollary 6.1. *Let X be a smooth, proper K -variety, which has potential good reduction after a base change of order q^r , $q \neq \text{char}(k)$ a prime. Then*

$$\chi(X) := \sum_{i \geq 0} (-1)^i \dim H^i(X_{K^s}, \mathbb{Q}_l) = s(X) \pmod{q}$$

with K^s a separable closure of K , $l \neq \text{char}(k)$ a prime. In particular, if $\chi(X)$ does not vanish modulo q , then X has a K -rational point.

Proof. Let L/K be the field extension of degree q^r , such that there is a smooth and proper model of X_L . Let \mathcal{O}_L be the ring of integers of L , $T := \text{Spec}(\mathcal{O}_L)$, and $\varphi : \mathcal{Y} \rightarrow T$ a smooth and proper model of X_L , which is in particular a weak Néron model of X_L . So by definition $s(X_L) = \chi_c(\mathcal{Y}_k)$. As φ is proper, \mathcal{Y}_k is proper over k , and hence the ordinary cohomology coincides with the cohomology with proper support, i.e. $\chi_c(\mathcal{Y}_k) = \chi(\mathcal{Y}_k)$. As φ is proper and smooth, by [Del77, Exposé V, Theorem 3.1] we get bijections between $H^i(\mathcal{Y}_k, \mathbb{Z}/n\mathbb{Z})$, and $H^i(\mathcal{Y}, \mathbb{Z}/n\mathbb{Z})$, and $H^i(X_L \times_{\text{Spec}(L)} \text{Spec}(L^s), \mathbb{Z}/n\mathbb{Z})$ for all i , with L^s a separable closure of L . Therefore we have for all i that

$$\dim H^i(\mathcal{Y}_k, \mathbb{Q}_l) = \dim H^i(X_L \times_{\text{Spec}(L)} L^s, \mathbb{Q}_l).$$

Note that $L^s = K^s$, because L/K is a tame Galois extension. Therefore we get $X_L \times_{\text{Spec}(L)} \text{Spec}(L^s) = X \times_{\text{Spec}(K)} \text{Spec}(K^s) = X_{K^s}$, hence

$$\chi(X) = \sum_{i \geq 0} (-1)^i \dim H^i(X_{K^s}, \mathbb{Q}_l) = \sum_{i \geq 0} (-1)^i \dim H^i(\mathcal{Y}_k, \mathbb{Q}_l) = \chi(\mathcal{Y}_k).$$

This implies that $s(X_L) = \chi(X)$. Hence Theorem 5.4 implies the corollary. \square

Corollary 6.2. *Let X be a smooth, proper K -variety, let L/K be a tame Galois extension of prime order q , and assume that there is a smooth and proper model of X_L with a good $G := \text{Gal}(L/K)$ -action extending the Galois action on X_L , i.e. in particular X has potential good reduction after a base change of order q . If $\chi(X, \mathcal{O}_X)$ does not vanish modulo q , then X has a K -rational point.*

Proof. Let \mathcal{O}_L be the ring of integers of L , $T := \text{Spec}(\mathcal{O}_L)$. Let $\varphi : \mathcal{Y} \rightarrow T$ be the smooth and proper model of X_L on which there is a good G -action extending the Galois action on X_L . As $X_L \subset \mathcal{Y}$ is G -invariant, the same holds for $\mathcal{Y}_k \subset \mathcal{Y}$, hence the G -action on \mathcal{Y} restricts to a good G -action on \mathcal{Y}_k . Let $f : \mathcal{Y}_k \rightarrow \mathcal{Y}_k/G$ be the quotient.

Assume that the action of G on \mathcal{Y} has no fixed point, so in particular the action of G on \mathcal{Y}_k has no fixed point. As q is a prime, the action is free. So f is a finite, étale morphism of degree q by [Gro63, Exposé V, Corollaire 2.3]. Moreover \mathcal{Y}_k is smooth and proper over k , because φ is smooth and proper. As f is étale and finite, \mathcal{Y}_k/G is smooth and proper over k , too. As f is étale, $f^*(T_{\mathcal{Y}_k/G}) = T_{\mathcal{Y}_k}$, and therefore $f^*(\text{td}(T_{\mathcal{Y}_k/G})) = \text{td}(T_{\mathcal{Y}_k})$. Let $s : \mathcal{Y}_k \rightarrow \text{Spec}(k)$ and $s' : \mathcal{Y}_k/G \rightarrow \text{Spec}(k)$ be the structure maps. We have $s = s' \circ f$. Using [Ful98, Corollary 15.2.2], and the projection formula in the third line, we obtain

$$\begin{aligned} \chi(\mathcal{Y}_k, \mathcal{O}_{\mathcal{Y}_k}) &= s_*(\text{ch}(\mathcal{O}_{\mathcal{Y}_k}) \text{td}(T_{\mathcal{Y}_k})) \\ &= s'_*(f_*(\text{ch}(\mathcal{O}_{\mathcal{Y}_k}) f^*(\text{td}(T_{\mathcal{Y}_k/G})))) \\ &= s'_*(f_*(\text{ch}(\mathcal{O}_{\mathcal{Y}_k})) \text{td}(T_{\mathcal{Y}_k/G})) \\ &= s'_*(\deg(f) \text{ch}(\mathcal{O}_{\mathcal{Y}_k/G}) \text{td}(T_{\mathcal{Y}_k/G})) = q \chi(\mathcal{Y}_k/G, \mathcal{O}_{\mathcal{Y}_k/G}). \end{aligned}$$

This implies that $\chi(\mathcal{Y}_k, \mathcal{O}_{\mathcal{Y}_k}) = 0 \pmod{q}$.

Note that $H^i(X_L, \mathcal{O}_{X_L}) = H^i(X_L, \mathcal{O}_X \otimes_K L) = H^i(X, \mathcal{O}_X) \otimes_K L$ holds for all $i \geq 0$. So in particular $\chi(X_L, \mathcal{O}_{X_L}) = \chi(X, \mathcal{O}_X)$. As φ is smooth and proper, and T is connected, by [Gro61, Theorem 7.9.4.I] the Euler characteristic is constant on the fibers of φ , hence $\chi(\mathcal{Y}_k, \mathcal{O}_{\mathcal{Y}_k}) = \chi(X_L, \mathcal{O}_{X_L})$, which does not vanish modulo q . This is a contradiction, hence $\mathcal{Y}^G \neq \emptyset$. So Corollary 4.4 implies that X has a K -rational point. \square

7. APPENDIX

In this section we show two lemmas concerning tame cyclic actions on Henselian, regular, local rings. These results are used in Section 4.

The following lemma should be known to the experts; a similar statement can be found in [Ser68].

Lemma 7.1. *Let A be a regular, Henselian ring of dimension n with maximal ideal \mathfrak{m} , such that its residue field κ is a field of $\text{char}(\kappa) \nmid r$ containing all r -th roots of unity, and let $\alpha \in \text{Aut}(A)$ with $\alpha^r = \text{id}$, such that the residual map on κ is trivial. There exists a regular system of parameters $x_1, \dots, x_n \in \mathfrak{m} \subset A$ with $\alpha(x_i) = \mu^{\ell_i} x_i$, $\mu \in A$ a primitive r -th root of unity, and $\ell_i \in \{0, \dots, r-1\}$. If there are $z_1, \dots, z_s \in \mathfrak{m} \subset A$, such that the $\bar{z}_1, \dots, \bar{z}_s \in \mathfrak{m}/\mathfrak{m}^2$ are linearly independent, and such that $\alpha(z_i) = \mu^{\ell_i} z_i$ for some $\ell_i \in \{0, \dots, r-1\}$, then we may chose $x_i = z_i$ for $i \leq s$.*

Proof. Consider the polynomial $p(x) := x^r - 1 \in A[x]$. Let $\mu \in \kappa$ be an r -th root of unity, hence $p(\mu) = 0 \in \kappa$. $p'(\mu) = r\mu^{r-1} \neq 0 \in \kappa$, because $r \neq 0 \in \kappa$. As A is Henselian, Hensel's Lemma gives us a $\tilde{\mu} \in A$, such that $\tilde{\mu} = \mu \pmod{\mathfrak{m}}$, and $p(\tilde{\mu}) = 0$, i.e. $\tilde{\mu}$ is a lift of μ , and $\tilde{\mu}^r = 1$. So we may fix a primitive r -th root of unity $\mu \in A$. Identify μ with its image in κ under the residue map.

As A is a regular local ring of dimension n with residue field κ , $\mathfrak{m}/\mathfrak{m}^2$ is an n -dimensional κ -vector space. Let $\bar{\alpha} \in \text{Aut}(\mathfrak{m}/\mathfrak{m}^2)$ be the automorphism induced by α . As the morphism on $A/\mathfrak{m} = \kappa$ induced by α is trivial, $\bar{\alpha}$ is a κ -linear map. For some algebraic closure $\bar{\kappa}$ of κ , there exists a basis of $\mathfrak{m}/\mathfrak{m}^2 \otimes_{\kappa} \bar{\kappa}$, such that the matrix corresponding to $\bar{\alpha}$ has Jordan normal form. As $\bar{\alpha}^r = \text{id}$, all eigenvalues are r -th roots of unity, i.e. powers of μ , and as $r \neq 0 \in \kappa \subset \bar{\kappa}$, the matrix is already diagonal. But all r -th roots of unity are assumed to be in κ , so $\bar{\alpha}$ is diagonalizable, too. Therefore $\mathfrak{m}/\mathfrak{m}^2$ decomposes into eigenspaces E_j . By assumption, for all i there exists a j such that $\bar{z}_i \in E_j$. Note that for all j one can choose a basis B_j of E_j such that for all i , $\bar{z}_i \in \cup B_j$. This uses the fact that the \bar{z}_i are linearly independent. Set $\{\bar{x}_{s+1}, \dots, \bar{x}_n\} := \cup B_j \setminus \{\bar{z}_1, \dots, \bar{z}_s\}$. As the E_j are eigenspaces, we have $\bar{\alpha}(\bar{x}_i) = \mu^{\ell_i} \bar{x}_i$ for some $\ell_i \in \{0, \dots, r-1\}$. Choose $\tilde{x}_i \in A$ such that $\tilde{x}_i \bmod \mathfrak{m}^2 = \bar{x}_i$. As r is invertible in A , we can define x_i for $i > s$ as follows.

$$x_i := \frac{1}{r} \sum_{j=0}^{r-1} \mu^{-\ell_i j} \alpha^j(\tilde{x}_i)$$

We have

$$\begin{aligned} \alpha(x_i) &= \frac{1}{r} \sum_{j=0}^{r-1} \mu^{-\ell_i j} \alpha^{j+1}(\tilde{x}_i) = \frac{\mu^{\ell_i}}{r} \sum_{j=0}^{r-1} \mu^{-\ell_i(j+1)} \alpha^{j+1}(\tilde{x}_i) \\ &= \frac{\mu^{\ell_i}}{r} \left(\sum_{j=1}^{r-1} \mu^{-\ell_i j} \alpha^j(\tilde{x}_i) + \mu^{-\ell_i r} \alpha^r(\tilde{x}_i) \right) = \mu^{\ell_i} x_i. \end{aligned}$$

Moreover

$$x_i \bmod \mathfrak{m}^2 = \frac{1}{r} \sum_{j=0}^{r-1} \mu^{-\ell_i j} \mu^{\ell_i j}(\bar{x}_i) = \frac{1}{r} \sum_{j=0}^{r-1} \bar{x}_i = \bar{x}_i,$$

hence $\{z_1, \dots, z_s, x_{s+1}, \dots, x_n\}$ is a system of regular parameters in A . □

Remark 7.2. In fact we do not need to assume in Lemma 7.1 that A is Henselian, but only that all the r -th roots of unity in κ lift to A . The same is true in Remark 7.4 and Lemma 7.5.

If we do not assume that r is prime to $\text{char}(\kappa)$, Lemma 7.1 is wrong. To see this, look at the following example:

Example 7.3. Let κ be an algebraically closed field with $\text{char}(\kappa) = 2$. Then $A := \kappa[[x, y]]$ is a complete local ring with maximal ideal $\mathfrak{m} = (x, y) \subset A$. Let $\alpha \in \text{Aut}(A)$ with $\alpha(P(x, y)) = P(x, x + y)$ for all $P(x, y) \in A$. We have that $\alpha^2(P(x, y)) = P(x, 2x + y) = P(x, y)$, because $\text{char}(\kappa) = 2$, hence $\alpha^2 = \text{id}$. Note that $\bar{\alpha} : \mathfrak{m}/\mathfrak{m}^2 \rightarrow \mathfrak{m}/\mathfrak{m}^2$ is not diagonalizable.

Let A be a ring, $\alpha \in \text{Aut}(A)$ with $\alpha^r = \text{id}$. Then α defines an action of $G := \mathbb{Z}/r\mathbb{Z}$ on A , and the subring

$$A^G := \{a \in A \mid \alpha(a) = a\} \subset A$$

is called the *ring of invariants*.

Remark 7.4. Let A be as in Lemma 7.1. Then $\mathfrak{m}^G := \mathfrak{m} \cap A^G \subset A^G$ is an ideal and we have $A^G/\mathfrak{m}^G \hookrightarrow A/\mathfrak{m} = \kappa$. With a proof similar to the proof of Lemma 7.1, we can show that there exists a lift $\tilde{s} \in A$ of $s \in \kappa$ such that $\alpha(\tilde{s}) = \tilde{s}$, i.e. $\tilde{s} \in A^G$, and hence $A^G/\mathfrak{m}^G = \kappa$. Hence there is a ringhomomorphism $A^G \rightarrow \kappa$, and hence we may consider $\kappa \otimes_{A^G} A$.

Note that $G := \mathbb{Z}/r\mathbb{Z}$ acts on $\kappa \otimes_{A^G} A$ given by $\text{id} \otimes \alpha \in \text{Aut}(\kappa \otimes_{A^G} A)$, such that the canonical maps $\rho_1 : A \rightarrow \kappa \otimes_{A^G} A$ and $\rho_2 : \kappa \rightarrow \kappa \otimes_{A^G} A$ are G -equivariant for this G -action, and the given G -action on A , and the trivial G -action on κ , respectively.

Lemma 7.5. *Let A be as in Lemma 7.1. Then $\kappa \otimes_{A^G} A \cong \kappa[x_1, \dots, x_m]/\mathfrak{I}$, $m \leq n$, and*

$$(\text{id} \otimes \alpha)(p(x_1, \dots, x_m)) = p(\mu^{\ell_0} x_1, \dots, \mu^{\ell_m} x_m)$$

for some $\ell_i \in \{1, \dots, r-1\}$, $p(x_1, \dots, x_m) \in \kappa \otimes_{A^G} A$, $\mu \in \kappa$ a primitive r -th root of unity, and $\mathfrak{I} \subset \kappa[x_1, \dots, x_m]$ is the ideal generated by monomials of the form $x_1^{s_1} \dots x_m^{s_m}$ with $s_1 \ell_1 + \dots + s_m \ell_m = sr$, $s \in \mathbb{N}$.

Proof. Set $\tilde{A} := \kappa \otimes_{A^G} A$. Consider $\rho_1 : A \rightarrow \tilde{A}$; $a \mapsto 1 \otimes a$. Take any $s \in \kappa$ and $a \in A$. As above we can choose a lift $\tilde{s} \in A^G$ of s . Hence we get $\rho_1(\tilde{s}a) = 1 \otimes \tilde{s}a = s \otimes a$, hence ρ_1 is surjective. Note that $0 = \rho_1(a) = 1 \otimes a$ for some $a \in A$ if and only if we can write $a = a_1 a_2$ for some $a_1 \in A^G$, $a_2 \in A$, and $r^G(a_1) = 0$, i.e. $a_1 \in \mathfrak{m}^G := \mathfrak{m} \cap A^G$, hence $\ker(\rho_1) = A \mathfrak{m}^G$.

By Lemma 7.1 there exists a system of parameters $y_1, \dots, y_n \in A$ such that $\alpha(y_i) = \tilde{\mu}^{\ell_i} y_i$, $\ell_i \in \{0, \dots, r-1\}$, $\tilde{\mu} \in A$ a primitive r -th root of unity, which is a lift of $\mu \in \kappa$. So $A \mathfrak{m}^G \subset A$ is the ideal generated by monomials of the form $y_1^{s_1} \dots y_n^{s_n}$ with $s_1 \ell_1 + \dots + s_n \ell_n = sr$, $s \in \mathbb{N}$. As \mathfrak{m}^{nr} is generated by monomials of degree nr in the y_i , all generators are divisible by y_i^r for at least one i . Note that for all i , $y_i^r \in \mathfrak{m}^G$. Hence $\mathfrak{m}^{nr} \subset A \mathfrak{m}^G$.

Set $N := nr$. So $\tilde{A} \cong \kappa \otimes_{A^G} (A/\mathfrak{m}^N)$. We show by induction that this is generated as a κ -algebra by the images of the y_i . The induction assumption is clear, because in this case $\kappa \otimes_{A^G} (A/\mathfrak{m}^1) \cong \kappa$. Assume that $\kappa \otimes_{A^G} (A/\mathfrak{m}^l)$ is generated as a κ -algebra by the images of the y_i . Let \tilde{A}_{l+1} be the subalgebra of $\kappa \otimes_{A^G} (A/\mathfrak{m}^{l+1})$ generated by the images of the y_i . Take any element $1 \otimes a$ in $\kappa \otimes_{A^G} (A/\mathfrak{m}^{l+1})$. By the induction assumption there is an $\tilde{a} \in A/\mathfrak{m}^{l+1}$, such that $a - \tilde{a} \in \mathfrak{m}^l/\mathfrak{m}^{l+1}$, and $1 \otimes \tilde{a} \in \tilde{A}_{l+1}$. Note that $\mathfrak{m}^l/\mathfrak{m}^{l+1}$ is a κ -vector space generated by monomials of degree l in the y_i . So $1 \otimes (\tilde{a} - a) \in \tilde{A}_{l+1}$, and therefore the same holds for $1 \otimes a = 1 \otimes \tilde{a} + 1 \otimes (a - \tilde{a})$. Altogether, \tilde{A} is generated as a κ -algebra by the images of the y_i .

Let x_1, \dots, x_m be the images of those y_i with $\ell_i \neq 0$. Note that, if $\ell_i = 0$, $y_i \in \mathfrak{m}^G \subset \ker(\rho_1)$, i.e. $\rho_1(y_i) = 0$. Hence the x_i generate \tilde{A} as a κ -algebra. Renumbering the y_i , we may assume that $\rho_1(y_i) = x_i$. We have

$$(\text{id} \otimes \alpha)(x_i) = (\text{id} \otimes \alpha)(1 \otimes y_i) = 1 \otimes \alpha(y_i) = 1 \otimes \tilde{\mu}^{\ell_i} y_i = \mu^{\ell_i} x_i.$$

Moreover, using $\ker(\rho_1) = A \mathfrak{m}^G$, we obtain that

$$\tilde{A} \cong \kappa[x_1, \dots, x_m]/\mathfrak{I}$$

with \mathfrak{I} generated by $x_1^{s_1} \dots x_m^{s_m}$ with $s_1 \ell_1 + \dots + s_m \ell_m = sr$, $s \in \mathbb{N}$. As $(\text{id} \otimes \alpha)$ is a κ -morphism, $(\text{id} \otimes \alpha)(p(x_1, \dots, x_m)) = p(\mu^{\ell_1} x_1, \dots, \mu^{\ell_m} x_m)$ with $\ell_i \in \{1, \dots, r-1\}$ for $p(x_1, \dots, x_m) \in \tilde{A}$. \square

Remark 7.6. Note that if A is of mixed characteristic, it is not a κ -algebra, but $A \otimes_{A^G} \kappa$ is. As we tensor over A^G , we keep the information concerning the G -action on A .

REFERENCES

- [BLR90] Bosch, Siegfried, Werner Lütkebohmert, and Michel Raynaud: *Néron models*, volume 21 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1990, ISBN 3-540-50587-3.
- [Del77] Deligne, Pierre: *Cohomologie étale*. Lecture Notes in Mathematics, Vol. 569. Springer-Verlag, Berlin, 1977. Séminaire de Géométrie Algébrique du Bois-Marie SGA 4 $\frac{1}{2}$, Avec la collaboration de J. F. Boutot, A. Grothendieck, L. Illusie et J. L. Verdier.

- [Edi92] Edixhoven, Bas: *Néron models and tame ramification*. *Compositio Math.*, 81(3):291–306, 1992, ISSN 0010-437X.
- [EN11] Esnault, Hélène and Johannes Nicaise: *Finite group actions, rational fixed points and weak Néron models*. *Pure Appl. Math. Q.*, 7(4, Special Issue: In memory of Eckart Viehweg):1209–1240, 2011, ISSN 1558-8599.
- [Ful98] Fulton, William: *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 1998, ISBN 3-540-62046-X; 0-387-98549-2.
- [Gro61] Grothendieck, Alexander: *Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes*. *Inst. Hautes Études Sci. Publ. Math.*, (8):222, 1961, ISSN 0073-8301.
- [Gro63] Grothendieck, Alexander: *Revêtements étales et groupe fondamental. Fasc. I: Exposés 1 à 5*, volume 1960/61 of *Séminaire de Géométrie Algébrique*. Institut des Hautes Études Scientifiques, Paris, 1963.
- [Gro67] Grothendieck, Alexander: *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV*. *Inst. Hautes Études Sci. Publ. Math.*, (32):361, 1967, ISSN 0073-8301.
- [GW10] Görtz, Ulrich and Torsten Wedhorn: *Algebraic geometry I*. *Advanced Lectures in Mathematics*. Vieweg + Teubner, Wiesbaden, 2010, ISBN 978-3-8348-0676-5. Schemes with examples and exercises.
- [Har77] Hartshorne, Robin: *Algebraic Geometry*. Springer-Verlag, New York, 1977, ISBN 0-387-90244-9. *Graduate Texts in Mathematics*, No. 52.
- [Liu02] Liu, Qing: *Algebraic geometry and arithmetic curves*, volume 6 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, 2002, ISBN 0-19-850284-2. Translated from the French by Reinie Ern  , Oxford Science Publications.
- [Nic12] Nicaise, Johannes. Private communication, 2012.
- [NS11a] Nicaise, Johannes and Julien Sebag: *Motivic invariants of rigid varieties, and applications to complex singularities*. In *Motivic Integration and its Interactions with Model Theory and Non-Archimedean Geometry, I, volume 383 of London Mathematical Society Lecture Notes Series*. Cambridge University Press, 2011.
- [NS11b] Nicaise, Johannes and Julien Sebag: *The Grothendieck Ring of varieties*. In *Motivic Integration and its Interactions with Model Theory and Non-Archimedean Geometry, I, volume 383 of London Mathematical Society Lecture Notes Series*. Cambridge University Press, 2011.
- [Ser68] Serre, Jean Pierre: *Groupes finis d’automorphismes d’anneaux locaux r  guliers*. In *Colloque d’Alg  bre (Paris, 1967), Exp. 8*, page 11. Secr  tariat math  matique, Paris, 1968.
- [Ser79] Serre, Jean Pierre: *Local fields*, volume 67 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1979, ISBN 0-387-90424-7. Translated from the French by Marvin Jay Greenberg.
- [Ser09] Serre, Jean Pierre: *How to use finite fields for problems concerning infinite fields*. In *Arithmetic, geometry, cryptography and coding theory*, volume 487 of *Contemp. Math.*, pages 183–193. Amer. Math. Soc., Providence, RI, 2009.